Twisted generalized cohomology and applications

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0. Global overview

Theme

- **Declaration:** All twisted generalized cohomology theories we consider (or even all explicitly constructed) are motivated to various degrees by physics.

Math *from* physics

Q1: *What new mathematical structures and constructions can we extract from studying physical models?*

Math *in* Physics

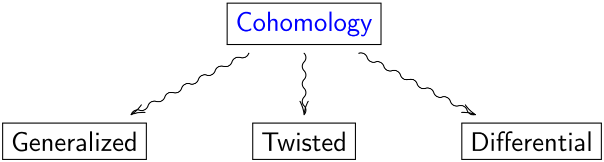
Q2: *What mathematical structures/conditions/tools should we have in place in order to properly define a physical theory?*

`By phrasing in context of physics, the math becomes more transparent`

Upshot

Generalities on what physics wants

Nontrivial physical entities, such as fields, charges, etc. generically take values in cohomology.



1. **Generalized**: Algebraically via formal groups or topologically via bundles.
2. **Twisting**: Symmetries via automorphisms.
3. **Differentially refined**: Include geometric data, such as connections, Chern character form, smooth structure, smooth representatives of maps ...

A motivation for generalized cohomology

Modelling of fields in physics, in particular quantum field theory, string theory and M-theory.

$$\begin{array}{ccc}
\Omega^\bullet(M) & & \textcolor{blue}{\textit{Classical}} \\
\Big\downarrow \text{exact complex} & & \Big\downarrow \\
H^\bullet_{\mathrm{dR}}(M) & & \\
\Big\downarrow \text{quantization} & & \\
H^\bullet(M;\mathbb{Z}) & & \\
\Big\downarrow \text{anomaly cancellation} & & \\
E^\bullet(M) & & \textcolor{blue}{\textit{Quantum}}
\end{array}$$

Twists

- We would like to introduce automorphisms.
- These arise from geometric and physical considerations.
- For the homotopy point of view: moduli/family setting; bundles of spectra.

$$\underset{\circlearrowright\, twist_{\Omega}}{\Omega^\bullet(M)} \xrightarrow[\text{complex}]{\text{exact}} \underset{\circlearrowright\, twist_{dR}}{H^\bullet_{\mathrm{dR}}(M)} \xrightarrow{\text{quantization}} \underset{\circlearrowright\, twist_{H}}{H^\bullet(M;\mathbb{Z})} \xrightarrow[\text{cancellation}]{\text{anomaly}} \underset{\circlearrowright\, twist_{E}}{E^\bullet(M)}$$

Relations among various twists?

Twisted de Rham cohomology

- The de Rham complex

$$(\Omega^\bullet, d): \quad \ldots \xrightarrow{d} \Omega^i(X) \xrightarrow{d} \Omega^{i+1}(X) \xrightarrow{d} \ldots$$

- Twist by a 1-form built out of scalar function: $d \rightsquigarrow d_\phi := d + d\phi\wedge$ with $d_\phi^2 = 0$.

Example (Witten's deformation of Morse theory)

For smooth $f : M \to \mathbb{R}$, the Witten differential is $d_s = e^{-sf} d e^{sf} = d + sdf\wedge$, where $s \in \mathbb{R}$. Then $d_s^2 = 0$, $d_s : \Omega^p \to \Omega^{p+1}$. The term e^{-sf} is a quasi-isomorphism

$$\begin{array}{ccccccc}
\ldots & \longrightarrow & \Omega^p & \xrightarrow{d} & \Omega^{p+1} & \longrightarrow & \ldots \\
 & & \downarrow{\scriptstyle e^{-sf}} & \circlearrowright & \downarrow{\scriptstyle e^{-sf}} & & \\
\ldots & \longrightarrow & \Omega^p & \xrightarrow{d_s} & \Omega^{p+1} & \longrightarrow & \ldots
\end{array}$$

and d_s yields isomorphic cohomology groups.

- Twist by a closed 3-form: $d_{H_3} = d - H_3\wedge$, with $d_{H_3}^2 = 0$.

Definition

Twisted de Rham cohomology: $H^i(X, H_3) := \frac{\ker(d_{H_3})}{\mathrm{im}(d_{H_3})}$

Example (*The Ramond-Ramond (RR) fields in string theory*)

$F = \sum_{i\leq 5} u^{-i} F_{2i+\epsilon}$, $\epsilon = 0$ or 1 for type IIA or type IIB string theory. These are twisted by a closed 3-form, the NS-field H_3.

- To make periodic: adjoin a generator u of degree 2 which implements the periodicity & makes total degree uniform:

$$\boxed{d_{H_3} = d - u^{-1} H_3\wedge}.$$

- In fact, one can build a differential by adding to d_H all expressions of the form $u^{-i}H_{2i+1}\wedge$, i.e.

$$d'_H = d + \sum_{i=0}^{\infty} u^{-i} H_{2i+1} \wedge \ .$$

- There is a twisted graded de Rham complex with differential $d + \sum_{i=1}^{\infty} u^{-i} H_{2i+1}\wedge$, provided the differential forms H_{2i+1} are closed.

Example (S.)

From string theory with $\mathcal{F} = F + *F$, where F is the Yang-Mills field and $*F$ its dual, one gets

$$\boxed{(d \text{ - } H_7\wedge)\mathcal{F} = 0}.$$

This gives a twisted differential $d_{H_7} = d - H_7\wedge$ which in nilpotent, i.e. squares to zero, $d^2_{H_7} = 0$, since H_7 is closed.

Twists of integral cohomology

- 1-d twists of integral cohomology given by a local system $Z \to M$.
- This is a bundle of groups isomorphic to $\mathbb{Z}$, so is determined by

$$H^1(M; Aut(\mathbb{Z})) \cong H^1(M; \mathbb{Z}_2)$$

  since the only nontrivial automorphism of $\mathbb{Z}$ is multiplication by -1.
- **Čech description.** Let $\{U_i\}$ be an open covering of M and $g_{ij} : U_i \cap U_j \to \{\pm 1\}$ a cocycle defining the local system Z. Then an element of $H^q(M; Z)$ is represented by a collection of q-cochains $a_i \in Z^q(U_i)$ which satisfy

$$a_j = g_{ij} a_i \quad \text{on } U_{ij} = U_i \cap U_j \ . \tag{1}$$

- Use a model as maps to Eilenberg-MacLane space $K(\mathbb{Z}, q)$.
  1. $K(\mathbb{Z}, 0) \simeq \mathbb{Z}$ on which -1 acts by multiplication.
  2. $K(\mathbb{Z}, 1) \simeq S^1$ on which -1 acts by reflection.
- Action of $Aut(\mathbb{Z})$ on $K(\mathbb{Z}, q)$ and the cocycle $g_{ij} \Rightarrow$ Associated bundle $\mathcal{H}^q \to M$ with fiber $K(\mathbb{Z}, q)$.
- Eq. (1) says that twisted cohomology classes are represented by sections of $\mathcal{H}^q \to M$. The twisted cohomology group $H^q(M; Z)$ is the set of homotopy classes of sections of $\mathcal{H}^q \to M$.

First main idea in a nutshell

Rational twisted cohomology arises as image of some Chern character.

Example

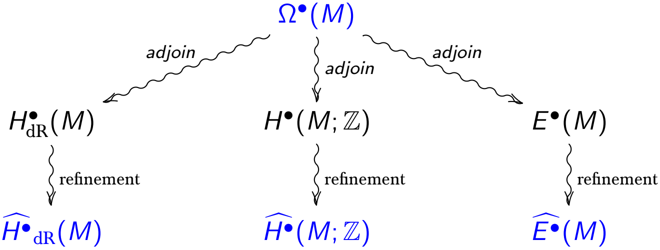
Degree three twist H_3:

$$\mathrm{ch}_{H_3} : \underbrace{K^\bullet(X, H_3)}_{\text{twisted K}-\text{theory}} \longrightarrow \underbrace{H^{\mathrm{ev}}(X, H_3)}_{twisted\ de\ Rham\ cohomology}$$

- Now if we are presented with higher degree twists on the left-hand-side, would they be images of some generalized Chern character whose domain is some generalized cohomology theory?

Differential refinement

We would like to introduce geometric data, say via differential forms. That is we would like to retain differential form representatives of cohomology classes.



Why differential cohomology?

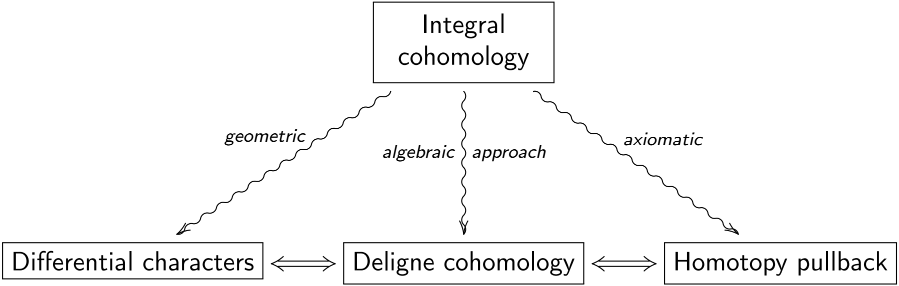
- Provides a way to refine secondary torsion invariants to $\mathbb{R}/\mathbb{Z}$-cohomology classes:
  - Chern-Simons invariants
  - Characteristic classes of flat vector bundles
  - invariants of elements in stable homotopy groups: e-invariant, f-invariant, $\cdots$
- Leads to results in geometry:

  e.g. [Chern-Simons] Differential Pontrjagin classes $\hat{p}_i$ show that $\mathrm{SO}(3) \cong \mathbb{R}P^3$ does not admit conformal immersion in $\mathbb{R}^4$.
- Provides a desirable setting for mathematical physics:

  Proper description of actions functionals in (topological) field theories.

Integral Cohomology - differential refinement

Three approaches to differential integral cohomology:



(i) Differential characters:

- A differential character of deg. k on a smooth M is a homomorphism

$$h : Z_{k-1}(M : \mathbb{Z}) \to U(1)$$

  from smooth integral-valued singular cocycles of degree $k-1$.

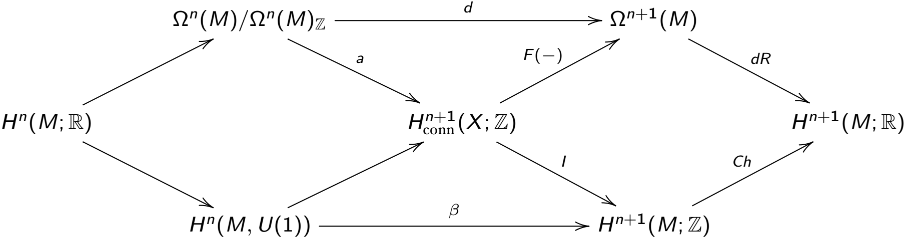
- Such that there exists a differential form $\mathrm{curve}(h) \in \Omega^k(M)$, uniquely determined by h and is called its curvature, such that

$$h(\partial c) = \exp\left(2\pi i \int_c \mathrm{curv}(h)\right) .$$

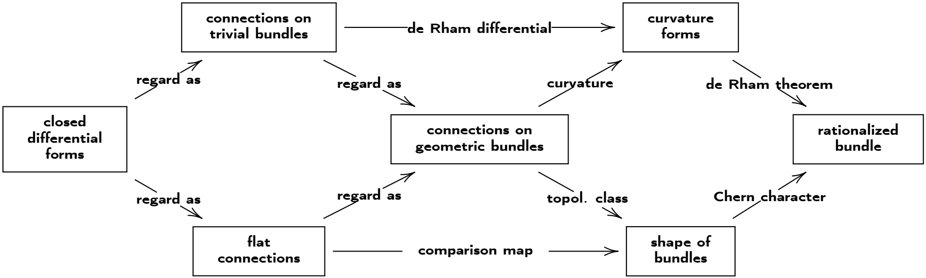
- Set of all differential characters on M of degree k is $\widehat{H}^k(M;\mathbb{Z})$. Pointwise multiplication provides an abelian group structure. There is also another multiplication $\widehat{H}^k(M;\mathbb{Z}) \times \widehat{H}^l(M;\mathbb{Z}) \longrightarrow \widehat{H}^{k+l}(M;\mathbb{Z})$ which turns this into a ring.

1. $\boxed{k=1}$: $U(1)$-valued functions. Given a $U(1)$-bundle with connection over M, one can associate a differential character by mapping any 1-cycle to the holonomy of the bundle along this cycle.
2. $\boxed{k>1}$: (higher) gerbes.

(ii) Axiomatic/Diagrammatic approach:



where d is the de Rham differential, F is the curvature map, I is the forgetful map, Ch is the rationalization, and β is the Bockstein associated with the exponential coefficient sequence. $0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{R} \xrightarrow{\exp} \mathbb{R}/\mathbb{Z} \longrightarrow 0.$



(iii) Deligne Cohomology:

- Consider the truncated de Rham complex

$$[\ \Omega^0 = \mathcal{O} \xrightarrow{d} \Omega^1 \xrightarrow{d} \Omega^2 \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n\ ]$$

- Replace the structure sheaf $\mathcal{O}$ with the multiplicative group $\mathcal{O}^\times$ under the exponential map to get the Deligne complex

$$[\ \mathcal{O}^\times \xrightarrow{d} \Omega^1 \xrightarrow{d} \Omega^2 \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n\ ]$$

- Deligne cohomology $H^{n+1}_{\mathcal{D}}(X)$ in degree $n+1$ is the hypercohomology for this complex of sheaves of abelian groups, i.e. abelian sheaf cohomology with coefficients in this chain complex.

- Note that

$$\underline{U(1)} = C^\infty(-, U(1)) .$$

Twisted + differential by combining Deligne with Axiomatic $\rightsquigarrow$ stacks: Part of long term project with Dan Grady.

If $\{U_\alpha\}$ is a good open cover of M, form the Čech-Deligne double complex

$$\begin{array}{ccccccccc}
\underline{\mathbb{Z}}(U_{\alpha_0\ldots\alpha_n}) & \xrightarrow{2\pi i} & \Omega^0(U_{\alpha_0\ldots\alpha_n}) & \xrightarrow{d} & \Omega^1(U_{\alpha_0\ldots\alpha_n}) & \xrightarrow{d} & \ldots & \xrightarrow{d} & \Omega^{n-1}(U_{\alpha_0\ldots\alpha_n}) \\
{\scriptstyle (-1)^{n-1}\delta}\uparrow & & {\scriptstyle (-1)^{n-1}\delta}\uparrow & & {\scriptstyle (-1)^{n-1}\delta}\uparrow & & & & {\scriptstyle (-1)^{n-1}\delta}\uparrow \\
\underline{\mathbb{Z}}(U_{\alpha_0\ldots\alpha_{n-1}}) & \xrightarrow{2\pi i} & \Omega^0(U_{\alpha_0\ldots\alpha_{n-1}}) & \xrightarrow{d} & \Omega^1(U_{\alpha_0\ldots\alpha_{n-1}}) & \xrightarrow{d} & \ldots & \xrightarrow{d} & \Omega^{n-1}(U_{\alpha_0\ldots\alpha_{n-1}}) \\
{\scriptstyle (-1)^{n-2}\delta}\uparrow & & {\scriptstyle (-1)^{n-2}\delta}\uparrow & & {\scriptstyle (-1)^{n-2}\delta}\uparrow & & & & {\scriptstyle (-1)^{n-2}\delta}\uparrow \\
\vdots & & \vdots & & \vdots & & & & \vdots \\
{\scriptstyle -\delta}\uparrow & & {\scriptstyle -\delta}\uparrow & & {\scriptstyle -\delta}\uparrow & & & & {\scriptstyle -\delta}\uparrow \\
\underline{\mathbb{Z}}(U_{\alpha_0\alpha_1}) & \xrightarrow{2\pi i} & \Omega^0(U_{\alpha_0\alpha_1}) & \xrightarrow{d} & \Omega^1(U_{\alpha_0\alpha_1}) & \xrightarrow{d} & \ldots & \xrightarrow{d} & \Omega^{n-1}(U_{\alpha_0\alpha_1}) \\
{\scriptstyle \delta}\uparrow & & {\scriptstyle \delta}\uparrow & & {\scriptstyle \delta}\uparrow & & & & {\scriptstyle \delta}\uparrow \\
\underline{\mathbb{Z}}(U_{\alpha_0}) & \xrightarrow{2\pi i} & \Omega^0(U_{\alpha_0}) & \xrightarrow{d} & \Omega^1(U_{\alpha_0}) & \xrightarrow{d} & \ldots & \xrightarrow{d} & \Omega^{n-1}(U_{\alpha_0})\ ,
\end{array} \tag{2}$$

where $U_{\alpha_0\alpha_1\ldots\alpha_k}$ denotes the k-fold intersection.

- The total operator on the double complex is the Čech-Deligne operator $D := d + (-1)^p\delta$, where d and δ is the de Rham and Čech differentials, respectively, acting on elements of degree p.
- The sheaf cohomology group $H^0(M; \mathcal{D}(n))$ can be identified with the group of diagonal elements $\alpha_{k,k}$ in the double complex which are Čech-Deligne closed $(d + (-1)^p\delta)\alpha_{k,k} = 0$, modulo those which are Čech-Deligne exact.

Proposition (Properties of Deligne cohomology)

Deligne cohomology satisfies the following properties:

(i) (Functoriality) For a smooth map between manifolds $M \to N$, we have an induced map

$$\widehat{H}^n(N;\mathbb{Z}) \to \widehat{H}^n(M;\mathbb{Z})\ .$$

(ii) (Additivity) For $M = \coprod M_\alpha$ a disjoint union of smooth manifolds, we have an isomorphism

$$\widehat{H}^n(M;\mathbb{Z}) \cong \bigoplus_\alpha \widehat{H}^n(M_\alpha;\mathbb{Z})\ .$$

(iii) (Mayer-Vietoris) For an open cover of M by open smooth manifolds U and V, we have a sequence

$$\dots \longrightarrow H^{*-2}(U \cap V;\mathbb{R}/\mathbb{Z}) \longrightarrow \widehat{H}^*(M;\mathbb{Z}) \longrightarrow \widehat{H}^*(U;\mathbb{Z}) \oplus \widehat{H}^*(U;\mathbb{Z}) \longrightarrow \widehat{H}^*(U \cap V;\mathbb{Z}) \longrightarrow H^{*+1}(M;\mathbb{Z}) \longrightarrow \dots\ .$$

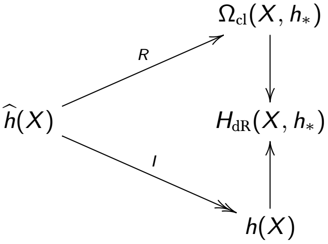
Mix between ordinary integral cohomology and cohomology with $\mathbb{R}/\mathbb{Z}$-coefficients.
Recall Diamond: Deligne cohomology is really a mixture of three different cohomology theories (integral, $\mathbb{R}/\mathbb{Z}$-coefficients, and de Rham) and captures the interactions between these theories.

Differential generalized cohomology

- Start with a generalized cohomology theory h
- $\Omega(X, h_*) := \Omega(X) \otimes_{\mathbb{Z}} h_*$ Smooth differential forms with coefficients in $h_* := h(*)$
- $\Omega_{\mathrm{cl}}(X, h_*) \subseteq \Omega(X, h_*)$ closed forms
- $H_{\mathrm{dR}}(X, h_*)$ cohomology of the complex $\big(\Omega(X, h_*), d\big)$

Def. A smooth extension of h is a contravariant functor

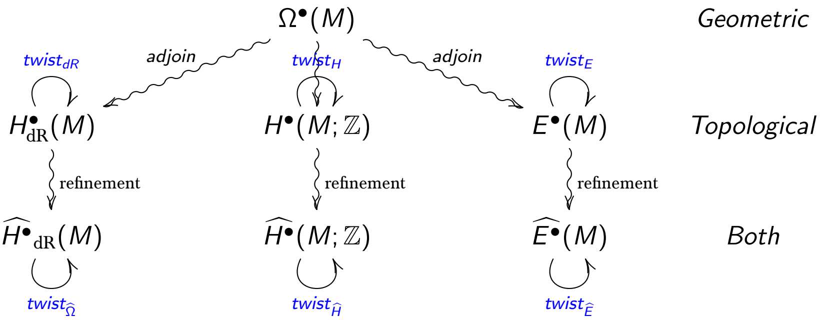
$\widehat{h}$: **Compact Smooth Manifolds** $\longrightarrow$ **Graded Abelian Grps**



[Chern-Simons, Cheeger-Simons, Simons-Sullivan, Hopkins-Singer, Bunke-Schick]

Full structure

Twisted ∩ Differential ∩ Generalized



Examples

1. With Craig Westerland & John Lind: Morava K-theory/E-theory, K-theories of *n*-vector bundles.
2. With Dan Grady: Differential refinements of cohomology theories including above.

Orientations

Orientation

$$\langle \text{cohom}\ ,\ [M]_{\text{hom}} \rangle \in \mathbb{C}, \mathbb{R}, \mathbb{Z}, \mathbb{Z}_2, \cdots\ .$$

- Disk neighborhood U of a pt m in M^n.
- Map $\varepsilon^{m,U} : M^n \to S^n$ collapse complement of $U \subset M$.
- $\pi_0(E) = \widetilde{E}_0(S^0) \cong \widetilde{E}_n(S^n) = E_n(S^n, *)$
  $1 \longrightarrow s_n$, canonical orientation of S^n.

Definition

$[M]_E \in E_n(M^n)$ is an E-orientation of M if, for all m, U,

$$\varepsilon_*^{m,U}[M]_E = \pm s_n\ .$$

- E-orientation $\Leftrightarrow$ existence of a *Thom class* u of TM (or NM).
- Thom isomorphism: $E^*(M) \xrightarrow{\cong} E^*(\mathrm{Th}^\eta)$, $x \mapsto x \cup u$
  - η ($O(n)$ metric space-) vector bundle on M.
  - Th^η Thom space ('one-pt compactification'):

$$\mathrm{Th}(V) := D(V)/S(V).$$

Note $\mathbb{D}^n/S^{n-1} \overset{\text{homeom}}{\simeq} S^n$.

Classifying spaces

A **classifying space** for the group G is a connected topological space BG, together with a principal G-bundle $EG \to BG$ such that

$$\boxed{\mathrm{Prin}_G(X) \cong [X, BG]}$$

The **Universal principal G-bundle**

$$\begin{array}{ccc} P = f^*(EG) & & EG \\ \downarrow & & \downarrow \\ X & \xrightarrow{\quad f \quad} & BG \end{array}$$

Example: $Prin_G(S^n) \cong \pi_n(BG)$

Properties:

1. Let $E \to B$ be a principal G-bundle with the property that the total space of E is *contractible*. Then (B, E) is a classifying space for G.
2. For *any* Lie group G there exists a classifying space BG.
3. Each nonzero class in $H^*(BG)$ *is* a universal characteristic class for principal G-bundles.

Obstructions to orientations are given by *characteristic classes*:

- Pontrjagin classes p_i and Stiefel-Whitney classes w_i in the real case,
- Chern classes c_i in the complex case.

The vanishing of one or more specific characteristic classes amounts to the ability to define and erect a desirable geometric and topological structure on the space.

Recall

$$\mathbb{Z}_2 \to Spin \to SO$$

Needed to define spinors (fermions), Dirac equation etc.

Perspectives:

1. Double cover of manifolds.
2. Part of short exact sequence (SES) of Lie groups.
3. Central extension of Lie groups.
4. Principal bundle of manifolds.
5. Fibration sequence of topological spaces.

- We would like to generalize this. For that, write $\mathbb{Z}_2 = K(\mathbb{Z}_2, 0)$, $U(1) = S^1 = K(\mathbb{Z}, 1)$ *Eilenberg-MacLane space*.
- Consider higher degree: $K(\mathbb{Z}, n)$ for $n > 1$.

$$\boxed{\pi_i K(\mathbb{Z}, n) = \begin{cases} \mathbb{Z} & \text{if } i = n \\ 0 & \text{otherwise.} \end{cases}}$$

The homotopy groups of the orthogonal group

The homotopy groups of the orthogonal group $O(n)$, for n sufficiently large, are

$$\pi_k\left(O(n)\right) = \begin{cases} \mathbb{Z}_2 & \text{for } k = 0, 1 \text{ mod } 8 \\ \mathbb{Z} & \text{for } k = 3, 7 \text{ mod } 8 \\ 0 & \text{otherwise} \end{cases} . \tag{3}$$

The condition on n is best understood by considering the *stable orthogonal group*, also know as the infinite orthogonal group, which is defined as the direct limit of the sequence of inclusions

$$O(1) \subset O(2) \subset \cdots \subset O = \bigcup_{k=0}^{\infty} O(k). \tag{4}$$

k		0	1	2	3	4	5	6	7
$\pi_k(\mathrm{O}(n))$		$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
$\mathrm{O}(n)\langle k\rangle$	$\mathrm{O}(n)$	$\mathrm{SO}(n)$	$\mathrm{Spin}(n)$		$\mathrm{String}(n)$				$\mathrm{Fivebrane}(n)$
		kill π_0	kill π_1		kill π_3				kill π_7

"kill" $\rightarrow$ "factorize".

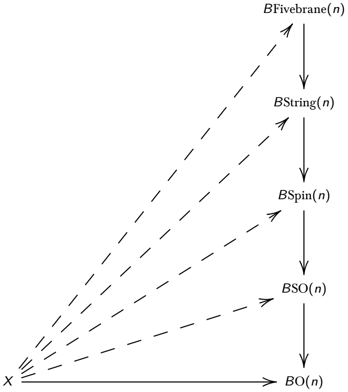
Example (String)

There is a fibration $K(\mathbb{Z},2) \rightarrow$ String $\rightarrow$ Spin. Take the corresponding long exact sequence on homotopy groups

$$\ldots \longrightarrow \pi_i(K(\mathbb{Z},2)) \longrightarrow \pi_i(\mathrm{String}) \longrightarrow \pi_i(\mathrm{Spin}) \longrightarrow$$

$$\longrightarrow \pi_{i-1}(K(\mathbb{Z},2)) \longrightarrow \pi_{i-1}(\mathrm{String}) \longrightarrow \ldots\ .$$

From the known homotopy groups of the fiber and the base:
For $i = 3, \cdots, 6$ and $j = 7$ this gives $\pi_i(\mathrm{String}) = 0$ and $\pi_7(\mathrm{String}) \cong \mathbb{Z}$.
Effectively means that in going from Spin to String we have "killed" $\pi_3(\mathrm{Spin})$.



Fivebrane structure

String structure

Spin structure

Orientation

Riemannian

target space

Whitehead tower of $B\mathrm{O}(n)$

String structures

- Homotopy fibration seq.: $K(\mathbb{Z},2) \to \mathrm{String}(n) \to \mathrm{Spin}(n)$.
- Geometric description of $K(\mathbb{Z},2)$ as $PU(\mathcal{H})$, the projective unitary group on an infinite-dimensional Hilbert space $\mathcal{H}$. Canonical degree three class DD of $PU(\mathcal{H})$ bundles. In the operator algebra language this is called the Dixmier-Douady class.
- Classifying functor: $K(\mathbb{Z},3) \to B\mathrm{String}(n) \to B\mathrm{Spin}(n)$.

String structure on X:

1. The obstruction is: $\boxed{\frac{1}{2}p_1(X) \in H^4(X;\mathbb{Z})}$.

2. The set of lifts, i.e. the set of String structures for a fixed Spin structure is a torsor for a quotient of $H^3(X;\mathbb{Z})$.

Fivebrane structures

- Homotopy fibration: $K(\mathbb{Z}, 6) \to \mathrm{Fivebrane}(n) \to \mathrm{String}(n)$.
- Classifying functor: $K(\mathbb{Z}, 7) \to B\mathrm{Fivebrane}(n) \to B\mathrm{String}(n)$.

Proposition (SSS)

1. The obstruction is given by $\boxed{\frac{1}{6} p_2(X) \in H^8(X; \mathbb{Z})}$.

2. The set of lifts, i.e. the set of Fivebrane structures for a fixed String structure is a torsor for a quotient of the seventh integral cohomology group $H^7(X; \mathbb{Z})$.

Going higher [S.]

k		7	8	9	10	11	12
$\pi_k(\mathrm{O}(n))$		$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0
$\mathrm{O}(n)\langle k\rangle$	$\mathrm{String}(n)$	$\mathrm{Fivebrane}(n)$	$\mathrm{O}\langle 9\rangle(n)$	$\mathrm{O}\langle 10\rangle(n)$		$\mathrm{Ninebrane}(n)$	
	kill π_7	kill π_8	kill π_9		kill π_{11}		

The mod 8 (Bott) periodicity of the homotopy groups of the orthogonal group motivates the following for the corresponding G-structures:

1. Space $O\langle 9\rangle$ corresponds to a 'shift by 8' analog of orientation: 2-Orient.
2. Space $O\langle 10\rangle$ corresponds to a 'shift by 8' analog of Spin structure: 2-Spin.

- One usually starts with classifying spaces and then take the loop space to define the above groups.

Connected covers [S.-Schreiber-Stasheff, S.]

$$
\begin{array}{ccccc}
 & & \vdots & & \\
 & & \downarrow & & \\
K(\mathbb{Z},11) & \longrightarrow & BO\langle 13\rangle = \mathrm{BNinebrane} & & \\
 & & \downarrow & & \\
K(\mathbb{Z}/2,9) & \longrightarrow & BO\langle 11\rangle = \mathrm{B2\text{-}Spin} & \xrightarrow{\frac{1}{240}p_3} & K(\mathbb{Z},12) \\
 & & \downarrow & & \\
K(\mathbb{Z}/2,8) & \longrightarrow & BO\langle 10\rangle = \mathrm{B2\text{-}Orient} & \xrightarrow{\alpha_{10}} & K(\mathbb{Z}/2,10) \\
 & & \downarrow & & \\
K(\mathbb{Z},7) & \longrightarrow & BO\langle 9\rangle = \mathrm{BFivebrane} & \xrightarrow{\alpha_9} & K(\mathbb{Z}/2,9) \\
 & & \downarrow & & \\
K(\mathbb{Z},4) & \longrightarrow & BO\langle 8\rangle = \mathrm{BString} & \xrightarrow{\frac{1}{6}p_2} & K(\mathbb{Z},8) \\
 & & \downarrow & & \\
K(\mathbb{Z}/2,1) & \longrightarrow & BO\langle 4\rangle = \mathrm{BSpin} & \xrightarrow{\frac{1}{2}p_1} & K(\mathbb{Z},3) \\
 & & \downarrow & & \\
 & & BO\langle 2\rangle = \mathrm{BSO} & \xrightarrow{w_2} & K(\mathbb{Z}/2,2) \\
 & & \downarrow & & \\
 & & BO & \xrightarrow{w_1} & K(\mathbb{Z}/2,2)
\end{array}
\tag{5}
$$

I. Generalized cohomology

Generalized Cohomology Theories

- **CW**: the category whose objects are CW-complexes with a 0-cell chosen as a base-point and whose maps are basepoint preserving maps.
- **Ab**: the category of abelian groups.

Definition

A cohomology theory on **CW** is a sequence $\{E^n\}_{n \in \mathbb{Z}}$ of functors $E^n : \mathbf{CW}^{\mathrm{op}} \to \mathrm{Ab}$ together with natural isomorphisms $E^n(X) \cong E^{n+1}(\Sigma X)$ for all $X \in \mathbf{CW}$, such that the Eilenberg-Steenrod axioms are satisfied:

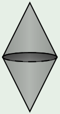
1. **Homotopy:** if $f, g : X \to Y$ are homotopic (preserving basepoints) then the induced maps $E^n(Y) \to E^n(X)$ are isomorphic.
2. **Inclusion:** For each inclusion $A \hookrightarrow X$ in **CW**, the sequence $E^n(X/A) \to E^n(X) \to E^n(A)$ is exact.
   Excision: U is contractible, inclusion $X - U \hookrightarrow X$ induces an iso in E.
3. **Additivity:** For a wedge sum $X = \bigvee_\alpha X_\alpha$ with inclusions $\iota_\alpha : X_\alpha \hookrightarrow X$, the product map $\prod_\alpha \iota_\alpha E^n(X) \xrightarrow{\cong} \prod_\alpha E^n(X_\alpha)$.
4. **Long exact sequence:** in cohomology for pairs of topological spaces (X, A):
$$\cdots \to E^n(X, A) \to E^n(X) \to E^n(A) \to E^{n+1}(X, A) \to \cdots$$
5. **Dimension:** $E^*(\mathrm{pt}) = \oplus_n E^n(\mathrm{pt})$ is a graded abelian group, *coefficient group*.

The Grothendieck construction K associates an abelian group to any semigroup by formally adjoining inverses.
$K(\mathbb{N}) = \mathbb{Z}$: Identify $(n, m) \sim (n + k, m + k)$ which we think of as $n - m$.

Example (K-theory K or KU)

- $\mathrm{Vect}(X)$: set of isomorphism classes of complex vector bundles over X. Abelian semigroup with op. $+$ coming from Whitney sum of vector bundles.
- $K^0(X)$: the universal group associated to $\mathrm{Vect}(X)$. Elements are formal differences of isomorphism classes of vector bundles.
- Given a map $f : X \to Y$ and vector bundles ξ over Y, we have pullback bundle $f^*\xi$ over X. Passing to isomorphism classes and formal differences, get $f^* : K^0(Y) \to K^0(X)$.
- Makes $K^0(-)$ into a *contravariant functor* from spaces to abelian groups.
- Reduced theory: $\widetilde{K} := \ker \mathrm{rank}$,

  $\mathrm{rank}\colon \mathrm{K} \to \mathbb{Z} \cong K(x_0)$
  [E] - [F] $\mapsto [E_{x_0}] - [F_{x_0}] = \mathrm{rank}_{x_0}$.
- Define $\widetilde{K}^{-n}(X)$ as $\widetilde{K}^0(\Sigma^n X)$ for $n \in \mathbb{N}$.



- $K^n(X) \cong K^{n-2}(X)$, $n \in \mathbb{Z}$, 2-periodic cohomology theory (*Bott*).
- $K^0(\mathrm{pt}) = \begin{cases} \mathbb{Z} & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$
- *Graded ring*: $K^*(\mathrm{pt}) = \mathbb{Z}[u, u^{-1}]$, $|u| = -2$.

Spectra

- A contravariant functor $H : \mathrm{Space} \to \mathrm{Sets}$ is called a homotopy functor if $H(f) = H(g)$ holds whenever $f \simeq g$.
- H is called representable if it is naturally isomorphic to the hom functor $\mathrm{Hom}(-, X)$ for some object X of Spaces.
- **Brown representability theorem:** H is representable iff H satisfies the additivity and Mayer-Vietoris axioms.

Definition

A spectrum is an object representing a generalized cohomology theory.
It is a collection of spaces h_n, one for each $n \in \mathbb{Z}$, such that

$$E^n(X) = [X, \Omega^\infty h_n]$$

together with connecting maps (inclusions) $\Sigma h_n \to h_{n+1}$ (note: $E^n(X) \cong E^{n+1}(\Sigma X)$).

Example (Cohomology)

Singular cohomology $H^n(-; A)$ is representable since $H^n(X; G) \cong [X, K(A, n)]$.
Eilenberg-MacLane spectrum HA with $HA_n = K(A, n)$, Eilenberg-MacLane space

$$\pi_i(K(A, n)) \cong \begin{cases} A, & i = n \\ 0, & i \neq n \end{cases}$$

Note: $C \cong K(C, 0)$ for C a discrete group, and $U(1) \cong K(\mathbb{Z}, 1)$.

Example (K-theory)

K-theory $K(X)$ is representable $K^0(X) \cong [X, BU \times \mathbb{Z}]$

$$BU = \text{classifying space of the stable unitary group as colimit } U = \varinjlim_n U(n)$$

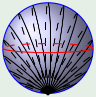
The spectrum is $\mathcal{U}$ with $\mathcal{U}_{2n} = BU \times \mathbb{Z}$ and $\mathcal{U}_{2n+1} = U$ (Bott periodicity).

A spectrum may be constructed out of a space. The suspension spectrum of a space X is a spectrum $X_n = S^n \wedge X$, with the structure maps being the identity.

Example (Sphere spectrum)

The sphere spectrum $\mathbb{S} = \Sigma^\infty S^0$ is the 'smallest nontrivial' spectrum.
It is the suspension spectrum of S^0 = set of two points.

- the nth space is the spectrum is $\Sigma^n S^0 = S^n$.
- the structure maps $\Sigma S^n \to S^{n+1}$ are the canonical homeomorphisms.



$\mathbb{S}$ is a unit for the smash product (suspension).
It is the spectrum for **cohomotopy**

$$\pi^n(X) = [X, \Sigma^\infty S^0]_n = [X, S^n] \,.$$

Note that for **homotopy**

$$\pi_n(X) = [\Sigma^\infty S^0, X]_n = [S^n, X] \,.$$

Ring spectra

Definition

A ring spectrum is a spectrum E such that the diagrams that describe the ring axioms in terms of smash product (smash=Cartesian/wedge) commute up to homotopy. There is a multiplication map

$$\mu : E \wedge E \to E$$

and a unit map $\eta : \mathbb{S} \to E$, where $\mathbb{S}$ is the sphere spectrum with $\mathbb{S}_n = S^n \cong \Sigma^n S^0$, such that

- Associative *up to homotopy*: $\mu(\mathrm{id} \wedge \mu) \sim \mu(\mu \wedge \mathrm{id})$.
- Unital *up to homotopy*: $\mu(\mathrm{id} \wedge \eta) \sim \mu(\eta \wedge \mathrm{id}) \sim \mathrm{id}$.

(Highly) structured:

- A_∞*-ring spectrum* (an algebra over an A_∞-operad) or
- E_∞*-ring spectrum* (an algebra over an E_∞-operad) by taking them as suitable (higher) loop spaces.

Examples

1. Both HR (R =ring) and $\mathcal{U}$ are E_∞.
2. **“Higher" examples**: (later)
   - Morava E-theory $E(n)$: E_∞
   - Morava K-theory $K(n)$: A_∞.

Omega-spectra

- A topological space is a loop space if it has a delooping. It is an infinite loop space if this delooping has itself a delooping, and so on.
- Infinite loop spaces are the grouplike E_∞ algebras in Top (grouplike E_∞ spaces).
- Consider the following category of spectra R: sequences of spaces R_n with homeomorphisms $R_n \to \Omega R_{n+1}$, whose zeroth spaces are infinite loop spaces:

$$E_0 \simeq \Omega E_1 \simeq \Omega^2 E_2 \simeq \cdots \simeq \Omega^\infty R_\infty \ .$$

- This defines a functor Ω^∞ from spectra to spaces. If the spectrum has a structure (E_∞ etc.) then that will be remembered by the space (E_∞-space etc.).
- The suspension spectrum functor Σ^∞ is the left adjoint.
- Adjunctions:

$$\mathrm{Top} \underset{}{\overset{(-)_+}{\rightleftarrows}} \mathrm{Top}_+ \underset{\Omega^\infty}{\overset{\Sigma^\infty}{\rightleftarrows}} \text{Spectra}$$

II. Twisted spectra

Units of ring spectra

- *Naively*: We want to classify twists. Classifying functor $B : \mathrm{Monoids} \to \mathrm{Top}_*$, adjoint to loop functor $\Omega : \mathrm{Top}_* \to \mathrm{Monoids}$. What to do if we have rings?

Example (Algebra)

Let $\mathrm{GL}_1 R$ denote the group of units of a commutative ring R.
The free abelian group functor $\mathbb{Z}$: Sets $\longrightarrow$ AbGrp induces a functor

$$\mathbb{Z} : \mathrm{Grp} \rightleftarrows \mathrm{Ring} : \mathrm{GL}_1$$

whose right adjoint is GL_1.
In particular, there is a natural map of rings $\mathbb{Z}[\mathrm{GL}_1 R] \longrightarrow R$.

- **Units of ring spectra** [May-Quinn-Ray]: Functor $\mathrm{GL}_1 : \mathrm{Rings} \to$ Group-like Spaces, $R \mapsto \mathrm{GL}_1(R) = R^\times$. If R is structured then so will be $\mathrm{GL}_1(R)$.
- Units $\mathrm{GL}_1(R)$ to be the union of the components in $\Omega^\infty R$ that correspond to group of units $(\pi_0 \Omega^\infty R)^\times \subseteq \pi_0(\Omega^\infty R)$ in the discrete ring $\pi_0 \Omega^\infty R$.

Definition

The unit is defined as the homotopy pullback diagram

$$\begin{array}{ccc}
\mathrm{GL}_1(R) & \longrightarrow & \Omega^\infty R = \underline{R}_0 \\
\downarrow & & \downarrow {\scriptstyle \pi_0} \\
(\pi_0 \Omega^\infty R)^\times & \xrightarrow{\subseteq} & \pi_0(\Omega^\infty R)
\end{array}$$

Twisted Spectra

[Ando-Blumberg-Gepner(-Hopkins-Rezk)]

- Can deloop as $\mathrm{GL}_1(R)$ is ∞-loop space: $\mathrm{GL}_1(R) \simeq \Omega^\infty(gl_1(R))$ if R E_∞ [Here $gl_1(R)$ is the spectrum of units of the spectrum R].
- Adjoint functor to GL_1 is the *infinite suspension functor* Σ_+^∞. Adjunction

$$\pi_0 \mathrm{Map}_{E_\infty/A_\infty}(\Sigma^\infty Z_+, R) \cong \pi_0 \mathrm{Map}_{E_\infty/A_\infty}(Z, \mathrm{GL}_1 R) \cong [BZ, B\mathrm{GL}_1 R]$$

  where Z is an E_∞ (resp. A_∞) space and $\mathrm{Map}_{E_\infty/A_\infty}$ denotes the space of E_∞ (resp. A_∞) maps (of spectra on the left and of spaces on the right).
- A module spectrum is a spectrum with an action of a ring spectrum (it generalizes a module in abstract algebra). For R an E_∞ or A_∞ ring spectrum, an R-module spectrum is a spectrum equipped with an R-action.
- To a space X and a maps $\tau : X \to B\mathrm{GL}_1 R$, one associates an R-module spectrum Th^τ representing twisted R-theory.
- The space $B\mathrm{GL}_1 R$ classifies the twists.

Definition

- The τ-twisted R-homology of X:

$$R_k(X)_\tau := \pi_0 \mathrm{Hom}_R(\Sigma^k R, \mathrm{Th}^\tau) \cong \pi_k X^\tau \ .$$

- The τ-twisted R-cohomology of X:

$$R^k(X)_\tau := \pi_0 \mathrm{Hom}_R(\mathrm{Th}^\tau, \Sigma^k R) \ .$$

Examples

There exist continuous maps

K-theory	$K(\mathbb{Z}, 3) \to BGL_1(K)$
Periodic de Rham cohom.	$K(\mathbb{R}, 2n+1) \to BGL_1(H\mathbb{R}[u, u^{-1}])$
Topological modular forms	$K(\mathbb{Z}, 4) \to BGL_1(TMF)$

where u is the periodicity element and $K(G, n)$ is an *Eilenberg-MacLane* space (whose sole homotopy group is G in dimension n).

- Since these spaces represent the cohomology functor $H^n(-, G)$, such cohomology classes give rise to twistings of the indicated generalized cohomology theory.

Bundles of spectra view: (schematic)

- Automorphism=symmetries
- We have a universal GL_1-bundle

$$\begin{array}{ccccc} P = \tau^*(E) & & E = E\mathrm{GL}_1(\mathcal{R}) & \longleftarrow & \mathrm{GL}_1(\mathcal{R}) \\ \downarrow & & \downarrow & & \\ M & \xrightarrow{\tau} & B\mathrm{GL}_1(\mathcal{R}) & & \end{array}$$

$\rightsquigarrow$ Parametrized spectra

Ex: Twisted K-theory (Rosenberg, Atiyah-Segal, ...)

- $R = \mathcal{U}$, $\mathrm{GL}_1(\mathbb{Z} \times BU) \cong \mathbb{Z}_2 \times BU = \mathbb{Z}^\times \times BU$.

  Pullback
  $$\begin{array}{ccc} \mathrm{GL}_1(\mathcal{U}) \cong \mathbb{Z}_2 \times BU & \longrightarrow & \mathbb{Z} \times BU \\ \downarrow & & \downarrow \\ \pi_0(\mathbb{Z} \times BU)^\times \cong \mathbb{Z}_2 & \xrightarrow{\subseteq} & \pi_0(\mathbb{Z} \times BU) \cong \mathbb{Z} \end{array}$$

- We have a universal GL_1-bundle

  $$\begin{array}{ccccc} P & & E\mathrm{GL}_1(\mathcal{U}) & \longleftarrow & \mathrm{GL}_1(\mathcal{U}) \\ \downarrow & & \downarrow & & \\ M & \xrightarrow{\tau} & B\mathrm{GL}_1(\mathcal{U}) & & \end{array}$$

- Twist: $$\begin{aligned} \tau : M \to B(\mathbb{Z}_2 \times BU_\otimes) & \cong B(\mathbb{Z}_2 \times BU(1) \times BSU_\otimes) \\ & \cong K(\mathbb{Z}_2, 1) \times K(\mathbb{Z}, 3) \times BBSU_\otimes \end{aligned}$$

  ($\otimes$ of virtual line bundles).

- Usually a degree three determinantal twist is isolated by postcomposing with the inclusion $i : K(\mathbb{Z}, 3) \to B\mathrm{GL}_1(\mathcal{U})$.

Twisted Chern character

- For a finite CW-complex X there is a ring isomorphism
$$\mathrm{ch} : K^{0/1} \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\sim} H^{\mathrm{ev}/\mathrm{odd}}(X;\mathbb{Q})$$
This is obtained by using $\mathrm{ch}(\mathcal{L}) = \exp(c_1(\mathcal{L}))$ for line bundles $\mathcal{L}$, the splitting principle, and additivity.
- At the level of spectra: There is a homotopy equivalence between the rationalized ring spectra $(\mathbb{Z} \times BU) \otimes_{\mathbb{Z}} \mathbb{Q}$ and the Eilenberg-MacLane spectrum $\prod_{n \geq 0} K(\mathbb{Q}, 2n)$.
- For a finite complex X, the groups $K^{0/1}(X;\tau) \otimes_{\mathbb{Z}} \mathbb{Q}$ are the homotopy groups of a bundle of spectra over X with fiber $\prod_{n \geq 0} K(\mathbb{Q}, 2n)$, which defines twisted rational cohomology.
- The splitting principle cannot be used for twisted K-theory classes, as they are usually represented by infinite-dimensional bundles.
- Nevertheless, one can construct a twisted version of the Chern isomorphism
$$\mathrm{ch}_\tau : K(X;\tau) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\sim} H_\tau^{\mathrm{ev}/\mathrm{odd}}(X;\mathbb{Q})$$

Twisted orientations

Twisted structures

Definition

Given a cocycle $\alpha_4 : M \to K(\mathbb{Z}, 4)$, an α-twisted String structure (or a String structure relative to α) on a Spin manifold M with classifying map $f : M \to B\mathrm{Spin}(n)$ is a homotopy η:

$$\begin{array}{ccc} M & \xrightarrow{\;\;f\;\;} & B\mathrm{Spin}(n) \\ & \searrow^{\alpha_4} \;\; \swarrow\!\!\swarrow_{\eta} & \Big\downarrow \tfrac{1}{2}p_1 \\ & & K(\mathbb{Z},4) \end{array} \quad . \tag{6}$$

Condition: $\boxed{\frac{1}{2}p_1(TX) + [\alpha_4] = 0}$. If α is trivial (i.e. factors through a point) then this reduces to ordinary String structure.

- For the Fivebrane: replace $\frac{1}{2}p_1$ with $\frac{1}{6}p_2$ and α_4 with β_8.

Theorem (Kriz-S.)

1. *A manifold X is orientable with respect to $\tilde{K}(2)$ if $W_7(X) = 0$.*
2. *A Spin manifold X is orientable with respect to $EO(2)$ if $w_4 = 0$. spaces.*

– Similar results hold also for Morava $E(2)$-theory when X is *Spin*.
– Proof involves the AHSS and study of the Milnor primitives in the Steenrod algebra.

Twisted TMF (and M-theory)

From various physical and mathematical considerations:

Conjecture (S.)

A twisted form of TMF describes the C-field in M-theory.

- Ando-Blumberg-Gepner indeed construct the spectrum *twisted TMF*.

Theorem (ABG)

(i) The spectrum twisted TMF exists.
(ii) Its orientation is a twisted String structure.
(iii) It admits a push forward and a Thom isomorphism.

Application:[S.] The charges of branes in M-theory take values in twisted TMF.

Generalized cohomology with degree seven twist

Conjecture (S.)

(i) Twisted Morava K-theory and Morava E-theory exit.
(ii) The first differential in their Atiyah-Hirzebruch spectral sequence (AHSS) corresponds to the cohomology class $W_7 + [H_7]$, where $[H_7]$ acts as the twist.

Theorem (S.-Westerland)

For mod 2 Morava K-theory, the set of homotopy classes of maps $K(\mathbb{Z}, n+2) \to BGL_1 K(n)$ is a group isomorphic to the 2-adic integers $\mathbb{Z}_2$. Furthermore, each component in the space of such maps is contractible.

This allows us to define, for any space X and class $H \in H^{n+2}(X)$, the twisted Morava K-theory $\boxed{K(n)^*(X; H)}$.

Structure	Condition	Generalized cohom.	Reference(s)
Spin	$w_2 = 0$	KO-theory	Atiyah-Bott-Shapiro
Spin^c	$W_3 = 0$	K-theory	Atiyah-Bott-Shapiro
Twisted Spin	$w_2 + [B_2] = 0$	Twisted KO-theory	Mathai-Murray-Stevenson
Twisted Spin^c	$W_3 + [H_3] = 0$	Twisted K-theory	Freed-Witten, Wang
String	$\frac{1}{2} p_1 = 0$	tmf	Ando-Hopkins-Strickland
Twisted String	$\frac{1}{2} p_1 + [G_4] = 0$	twisted tmf	Ando-Blumberg-Gepner
String$^{K(\mathbb{Z},3)}$	$W_7 = 0$	Morava $\widetilde{K}(2)$	Kriz-**S.**
Twisted String$^{K(\mathbb{Z},3)}$	$W_7 + [H_7] = 0$	Twisted Morava $\widetilde{K}(2)_H$	**S.**-Westerland
Membrane	$w_4 = 0$	Real Morava $EO(2)$	Kriz-**S.**
Twisted Membrane	$w_4 + [\alpha_4] = 0$	?	

For the last row:

Conjecture (S.)

Such a theory is a twisted form of real Morava $EO(2)$ at $p = 2$.

Work in progress with C. Westerland ...

Twisting chromatic theories

Formal Group Laws (FGLs)

Whitney sum: $c_1(\mathcal{L} \oplus \mathcal{L}') = c_1(\mathcal{L}) + c_1(\mathcal{L}')$ via splitting $[(1 + c_1(\mathcal{L}))(1 + c_1(\mathcal{L}'))]_{(2)}$

Tensor product: $c_1(\mathcal{L} \otimes \mathcal{L}') = c_1(\mathcal{L}) + c_1(\mathcal{L}')$ via cup product formula with $r = 1$

$c_r(\mathcal{L} \otimes E) = \sum_{i=0}^{r} c_1(\mathcal{L})^i c_{r-i}(E)$

Example (Complex line bundles in K-theory)

If $\mathcal{L}$ is a complex line bundle over a space $X \rightsquigarrow$ an element $[\mathcal{L}]$ of K-theory $K(X)$.
Define $c_1(\mathcal{L}) = [\mathcal{L}] - 1$ (normalization so that $c_1(\text{trivial } \ell.\text{ b.}) = 0$).
Then

$$\begin{aligned} c_1(\mathcal{L} \otimes \mathcal{L}') &= [\mathcal{L} \otimes \mathcal{L}'] - 1 \\ &= [\mathcal{L}][\mathcal{L}'] - 1 \qquad (K(X) \text{ is a commutative ring under } \otimes \text{ with unit } 1 = \underline{\mathbb{C}}) \\ &= c_1(\mathcal{L}) + c_1(\mathcal{L}') + c_1(\mathcal{L})c_1(\mathcal{L}') \\ &=: F(c_1(\mathcal{L}), c_1(\mathcal{L}')) \end{aligned}$$

Conditions on F:

1. Since $c_1(\text{trivial } \ell.\text{ b.}) = 0$, then $F(x, 0) = F(0, x) = x$.
2. $\otimes$ operation on complex ℓ.b.'s commutative up to isomorphism, so
$$F(x, y) = F(y, x).$$
3. $\otimes$ operation on complex ℓ.b.'s associative up to isomorphism, so
$$F(x, F(y, z)) = F(F(x, y), z).$$

Characterizing generalized cohomology I: *Formal groups*

$c_1(\mathcal{L}) : X \longrightarrow \mathbb{C}P^\infty = BU(1)$, classifying space.

Definition

A complex orientation on E is an element $c_1^E \in \widetilde{E}^2(\mathbb{C}P^\infty)$, the universal Chern class in E-theory, whose canonical restriction to S^2 is a unit $\pm 1 \in \widetilde{E}^2(S^2)$ via $S^2 = \mathbb{C}P^1 \hookrightarrow \mathbb{C}\mathrm{P}^\infty$.

Multiplication map classifying tensor product of line bundles

$$\mu : \mathbb{C}P^\infty \times \mathbb{C}\mathrm{P}^\infty \longrightarrow \mathbb{C}\mathrm{P}^\infty$$

Pulling back $\mu^*(z)$ get power series $F(x,y) \in E^*[[x,y]]$.

Definition

A formal group law (FGL) over a commutative ring R is a power series $F(x,y)$ with coefficients in R, such that

1. $F(x,0) = x,\ F(0,y) = y$;
2. $F(x,y) = x + y + \mathrm{H.O.T.} := x +_F y$;
3. $F(x(F(y,z)) = F(F(x,y),z)$.

$$c_1^E(\mathcal{L}_1 \otimes \mathcal{L}_2) = c_1^E(\mathcal{L}_1) +_F c_1^E(\mathcal{L}_2)$$

Examples

- *Additive FGL*: $F(x,y) = x + y$;
$$c_1^H(\mathcal{L}_1 \otimes \mathcal{L}_2) = c_1^H(\mathcal{L}_1) + c_1^H(\mathcal{L}_2).$$
- *Multiplicative FGL*: $F(x,y) = x + y + uxy$,
$$c_1^K(\mathcal{L}) = \pm 1 \pm [\mathcal{L}] \quad \text{so} \quad c_1^K(\mathcal{L}_1 \otimes \mathcal{L}_2) = c_1^K(\mathcal{L}_1) + c_1^K(\mathcal{L}_2) \pm c_1^K(\mathcal{L}_1)c_1^K(\mathcal{L}_2).$$

Classification: Over an algebraically closed field, every 1-dimensional algebraic group is isomorphic to either:

1. $\mathbb{G}_a$: the additive group.
2. $\mathbb{G}_m$: the multiplicative group.
3. $\mathcal{C}$: an elliptic curve.

FG(L)s are obtained by formal completion of the above groups (at the identity).

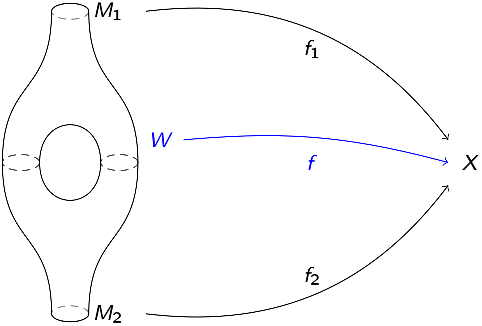
Definition

An elliptic spectrum is a spectrum which represents *elliptic cohomology*. A triple

1. Even periodic ring spectrum E;
2. an elliptic curve $\mathcal{C}$ over E_0;
3. an isomorphism of formal groups between $\mathrm{FGL}_{\mathcal{C}}$ and $\widehat{\mathcal{C}}$.

(Co)bordism

- M_i, $i = 1, 2$, smooth closed n-dimensional manifolds,
- $f_i : M_i \to X$ continuous maps.
- These maps are *bordant* if there is a map $f : W \to X$, with $\partial W = M_1 \coprod (-M_2)$, such that $f|_{M_i} = f_i$.



Definition

The map f is called a bordism between f_1 and f_2.

Complex cobordism

A *stable complex structure* on a real vector bundle E is a fiberwise complex structure on the Whitney sum $E \oplus \underline{\mathbb{R}}^k$.

Definition

The set of bordism classes of stably complex manifolds $MU_n(X)$ is a group under disjoint union, the *nth complex cobordism group of X*.

- M_* is a graded ring under *Cartesian products*:

$$MU_* \otimes MU_* \longrightarrow MU_*$$
$$[M] \otimes [N] \longmapsto [M \times N]$$

- From [Milnor, Novikov]: $MU_* = \mathbb{Z}[x_1, x_2, x_3, \cdots]$, $|x_i| = 2i$.
- The product $\otimes$ makes $MU^*(X)$ into a graded MU^*-algebra:
  - $MU^*(\mathbb{C}P^\infty) = MU^*[x]$, $|x| = 2$.

The two skeleton of $\mathbb{C}P^\infty \times \mathbb{C}P^\infty$ is just two spheres glued at a point. The two line bundles are tautological on one sphere and trivial on the other sphere

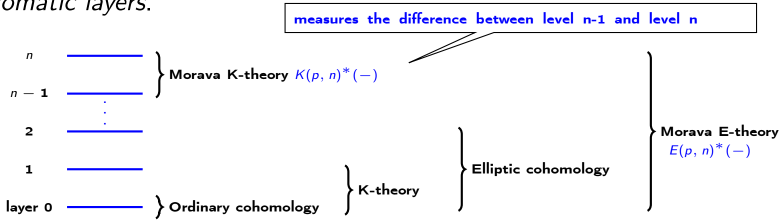
  - $MU^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \cong MU^*[x \otimes \mathbf{1} + \mathbf{1} \otimes x]$.
  - **As $\mathbb{C}P^\infty$ is a topological group, we have a product map $\mu : \mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$ inducing**

$$MU^*(\mathbb{C}P^\infty) \longrightarrow MU^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)$$
$$x \longmapsto \mu^*(x) = F(x \otimes \mathbf{1}, \mathbf{1} \otimes x)$$

Then F is a FGL over MU^*.

The chromatic viewpoint

- Systematic approach to information provided by different generalized cohomology theories is given by the chromatic filtration.
- Stable homotopy is the most desirable yet most complicated cohomology theory.
- Stable homotopy theory (localized at a prime) is naturally filtered by *chromatic layers*.



- nth layer (v_n periodic phenomena) $\leftrightarrow$ FGLs of height n $\rightsquigarrow$ n-dimensional varieties or higher genus curves.
- To a FG G of height n over an algebraically closed field k of char p $\overset{\text{associate}}{\rightsquigarrow}$ Morava K-theory.
- Closely related theory: Morava E-theory (**elliptic** for $n = 2$).

Introducing coefficients

- Work one prime at a time $\Rightarrow$ introduce coefficients such as:
  1. the prime fields $\mathbb{F}_p$.
  2. the p-local integers $\mathbb{Z}_{(p)} = \{\frac{a}{b} \in \mathbb{Q} | (p,b) = 1\}$.
  3. the p-adic integers $\mathbb{Z}_p$ (inverse limit of $\mathbb{Z}/p^n\mathbb{Z}$).
- Given a cohomology theory $E^*(-)$ and a ring $\mathcal{R}$, there is another cohomology theory $E\mathcal{R}(-) = E^*(-;\mathcal{R})$, E-cohomology with coefficients.

Examples

1. **Homology**: Replace chains $C_*(X)$ by $C_*(X) \otimes \mathcal{R}$.
2. **Cohomology**: Replace the cochain complex $\mathrm{Hom}(C_*(X), \mathbb{Z})$ by $\mathrm{Hom}(C_*(X), \mathcal{R})$.
3. Alternatively, replace dim. axiom by $H^*(\mathrm{pt};\mathcal{R}) = \begin{cases} \mathcal{R} & \text{if } n = 0, \\ 0 & \text{if } n \neq 0. \end{cases}$

- In general: Get universal coefficient spectral sequence relating $E\mathcal{R}^*(-)$ and $E^*(-)$.

Characterizing Generalized cohomology II: *Coefficients*

- There is a universal FGL F over a ring L (**Lazard ring**), such that for any commutative ring R and any FGL G over R, there is a unique homomorphism of rings $L \to R$ which carries F to G.
- **Quillen**: $L \cong MU_*$ ring of coefficients of complex cobordism.
- MU is a complex-oriented cohomology theory.
- Localizing at a prime p breaks MU into a direct sum of *Brown-Peterson* theories
  $BP_* = \mathbb{F}_p[v_1, v_2, v_3, \cdots], \qquad |v_n| = 2p^n - 2.$
- One inverts and/or kills regular sequences to get:

Examples

- *Johnson-Wilson* theory $BP\langle n\rangle_* = \mathbb{F}_p[v_1, \cdots, v_n]$
- Morava E-theories $E(n)_* = \mathbb{F}_p[v_1, \cdots, v_n, v_n^{-1}]$.
- Morava K-theories $K(n)_* = \mathbb{F}_p[v_n, v_n^{-1}]$.

These cohomology theories exist for each *prime* p and *chromatic level* n $\in \mathbb{N}$.

Twists of Morava $K(n)$ and $E(n)$

Recall: $K(n)_* = \mathbb{F}_p[v_n, v_n^{-1}]$, $|v_n| = 2p^n - 2$.
Study homotopy of space of maps $K(\mathbb{Z}, m) \to BGL_1 K(n)$.

- there are no nontrivial twistings by $H^m(X, \mathbb{Z})$ when $m \neq n+2$.
- For $m = n+2$ at any prime there are many such twistings:

Theorem (S.-Westerland)

For mod p Morava K-theory, the set of homotopy classes of maps $K(\mathbb{Z}, n+2) \to BGL_1 K(n)$ is a group isomorphic to the p-adic integers $\mathbb{Z}_p$.

A representative $u : K(\mathbb{Z}, n+2) \to BGL_1 K(n)$ of a particular topological generator of this group will be called the *universal* twisting. This allows us to define, for any space X and class $H \in H^{n+2}(X)$, the twisted Morava K-theory $\boxed{K(n)^*(X; H)}$.

- **Morava E-theory**: There is an A_∞-map $\pi : E(n) \to K(n)$ which gives $K(\mathbb{Z}, n+1) \xrightarrow{\varphi_n} GL_1 E(n) \xrightarrow{\pi} GL_1 K(n)$ as the twist.

Properties of twisted Morava K-theory

Twisted Morava K-theory satisfies (generalized) Eilenberg-Steenrod axioms plus the following basic properties, analogous to those of twisted K-theory:

Theorem (Properties of twisted Morava K-theory (SW))

Let $K(n)^*(X;H)$ be twisted Morava K-theory of a space X with twisting class H. Then:

1. (*Normalization*) If $H = 0$ then $K(n)^*(X;H) = K(n)^*(X)$.
2. (*Module property*) $K(n)^*(X;H)$ is a module over $K^0(n)(X)$.
3. (*Cup product*) There is a cup product homomorphism

$$K(n)^p(X;H) \otimes K(n)^q(X;H') \longrightarrow K(n)^{p+q}(X;H+H') ,$$

which makes $\oplus_H K(n)^*(X;H)$ into an associative ring (where H ranges over all of $H^{n+2}(X;\mathbb{Z})$).

4. (*Naturality*) If $f : Y \to X$ is a continuous map, then there is a homomorphism $f^* : K(n)^*(X;H) \to K(n)^*(Y;f^*H)$.

This also holds for the twisted Morava E-theory.

2-things

- *Vect* = **the category of finite-dimensional vector spaces**
- **2-vector spaces: Categorify, so base field $\mathbb{C} \rightsquigarrow$ a monoidal category.**
- **Vectors = k-tuples of scalars $\rightsquigarrow$ 2-vectors= k-tuples of vector spaces.**
- **Addition $+ \rightsquigarrow$ direct sum $\oplus$**
- **Multiplication $\times \rightsquigarrow$ tensor product $\otimes$**
  Example: $\mathbb{C}^k \rightsquigarrow Vect^k$.
- **Scalar multiplication:** $V \otimes \begin{pmatrix} V_1 \\ \vdots \\ V_k \end{pmatrix} = \begin{pmatrix} V \otimes V_1 \\ \vdots \\ V \otimes V_k \end{pmatrix}$ **Direct sum:** $\begin{pmatrix} V_1 \\ \vdots \\ V_k \end{pmatrix} \oplus \begin{pmatrix} W_1 \\ \vdots \\ W_k \end{pmatrix} = \begin{pmatrix} V_1 \oplus W_1 \\ \vdots \\ V_k \otimes W_k \end{pmatrix}$
- **2-linear map** $T : \mathbf{Vect}^k \to \mathbf{Vect}^1$ $\begin{pmatrix} T_{1,1} & \cdots & T_{1,k} \\ \vdots & & \vdots \\ T_{\ell,1} & \cdots & T_{\ell,k} \end{pmatrix} \begin{pmatrix} V_1 \\ \vdots \\ V_k \end{pmatrix} \oplus = \begin{pmatrix} \oplus_{i=1}^k T_{1,i} \otimes V_i \\ \vdots \\ \oplus_{i=1}^k T_{\ell,i} \otimes V_i \end{pmatrix}$
- **The natural transformations between these are matrices of linear transformations:**

$$\alpha = \begin{pmatrix} \alpha_{1,1} & \cdots & \alpha_{1,k} \\ \vdots & & \vdots \\ \alpha_{\ell,1} & \cdots & \alpha_{\ell,k} \end{pmatrix} : \begin{pmatrix} T_{1,1} & \cdots & T_{1,k} \\ \vdots & & \vdots \\ T_{\ell,1} & \cdots & T_{\ell,k} \end{pmatrix} \longrightarrow \begin{pmatrix} T'_{1,1} & \cdots & T'_{1,k} \\ \vdots & & \vdots \\ T'_{\ell,1} & \cdots & T'_{\ell,k} \end{pmatrix}$$

where each $\alpha_i j = T_{i,j} \to T'_{i,j}$ **is a linear map in the usual sense.**

These natural transformations give 2-morphisms between 2-linear maps, so that $\mathbf{Vect}^k$ is a bicategory with these as 2-cells:

$$\mathbf{Vect}^k \underset{T'}{\overset{T}{\Downarrow \alpha}} \mathbf{Vect}^1$$

2-algebra: 2-vector space L equipped with a bilinear functor $[-,-] : L \times L \to L$, the Lie bracket, that is skew-symmetric and satisfies the Jacobi identity up to ...

2-group: Monoidal (=$\otimes$) category G in which every morphism is invertible and every object has a weak inverse (xx^{-1} and $x^{-1}x$ are isomorphic to unit object.)

K-theory of 2-vector bundles [Baas-Dundas-Rognes]

- Algebraic K-theory (Quillen, Milnor, $\cdots$): doing linear algebra over a general ring $\overline{R}$ instead of over a field F. $\overline{R} \rightsquigarrow K_i(\overline{R})$.
- **Correspondence**: Given a discrete ring $\overline{R}$, the algebraic K-theory of the associated Eilenberg-MacLane spectrum $H\overline{R}$ reduces to Quillen's algebraic K-theory of rings $K(\overline{R})$.
- Usual rings: group $\mathrm{GL}_n(\overline{R}) = \mathrm{GL}_1(M_n(\overline{R}))$ via the ring of $n \times n$ matrices.
- $M_n(R)$ the ring spectrum whose mth space is built out of $\mathbf{n} = \{1, \cdots, n\}$ finite set and R_m.
- Inclusion of units: $B\mathrm{GL}_1(R) \to K(R) := \Omega B\Big(\coprod_{n \geq 0} B\mathrm{GL}_n(R)\Big)$
- Let ku be the connective complex K-theory spectrum.
- The underlying infinite loop space $\Omega^\infty ku = \mathbb{Z} \times BU$ of ku classifies virtual complex vector bundles.
- The algebraic K-theory spectrum $\boxed{K(ku)}$ has an interpretation as a Grothendieck group of 2-vector bundles.
- Idea: Form bundles with *Kapranov-Voevodsky 2-vector spaces* as fibers.
- Forming equivalence classes under morphisms leads to a bimonoidal category $2\mathrm{Vect}(X)$.
- The Grothendieck group completion of $2\mathrm{Vect}(X)$ is represented by the infinite loop space $\Omega^\infty K(ku)$ underlying the algebraic K-theory spectrum of ku.

K-theory of n-vector bundles [Lind-S.-Westerland]

- Considering the process of applying the functor $K(-)$ further,

Definition

The iterated algebraic K-theory spectrum

$$\mathrm{a}_n := K^{(n-1)}(ku) := \underbrace{K(K(\cdots K}_{n-1}(ku)\cdots))$$

- Should have an interpretation in terms of n-category theory.
- It is expected that algebraic K-theory in many cases increases chromatic complexity by one, i.e., that it produces a constant "red-shift" of one in stable homotopy theory [Ausoni-Rognes].
- In full generality, this is called the red-shift conjecture. Thus we expect that the spectrum we are studying is probing chromatic level n.

Relation to n-gerbes

Much as line bundles are the fundamental building blocks of vector bundles, $(n-1)$-gerbes are the simplest forms of n-vector bundles.

Example

$\boxed{n=2}$: *A 1-gerbe (a gerbe with band $U(1)$) gives rise to a rank one 2-vector bundle. At the level of classifying spaces*

$$K(\mathbb{Z},3) \to B\mathrm{GL}_1(ku) \to \Omega^\infty K(ku).$$

This is the 2-categorical analog of the map $\mathbb{C}P^\infty \to \mathbb{Z}\times BU$ representing the inclusion of line bundles into the Grothendieck group of vector bundles. The adjoint map $\Sigma^\infty \mathbb{C}P^\infty_+ \to ku$ is a map of E_∞ ring spectra.

- We study a family of analogous E_∞ ring maps

$$\boxed{\Sigma^\infty K(\mathbb{Z}, n+1)_+ \to \mathfrak{a}_n := K^{(n-1)}(ku)}$$

represents inclusion of $(n-1)$-gerbes into the Grothendieck group of n-vector bundles.

Twists of $K(A)$ from twists of A

[ABGHR]: Let h be a spectrum. Then a map $\tau : h \to \Sigma \mathrm{gl}_1 A$ may be regarded as an E_∞ *twisting of A*, in the sense that the infinite loop map

$$\tau : \Omega^\infty h \to B\,\mathrm{GL}_1\, A$$

allows us to twist the A-cohomology of a space X by an element of $H \in h^0(X)$, yielding twisted cohomology groups $A^*(X, H)$.

- E_∞ ring spectrum $A \Rightarrow K(A)$ is an E_∞ ring spectrum.
- There is a natural map $B\mathrm{GL}_1 A \to \Omega^\infty K(A)$ (*coming from the inclusion of A-lines into all cell A-modules*) and has image in $\mathrm{GL}_1\, K(A)$ (*since A-lines are invertible A-modules*), and in fact

  $$w : B\mathrm{GL}_1 A \to \mathrm{GL}_1 K(A)$$

  is an infinite loop map.

Proposition (LSW *)

For every E_∞-twisting $\tau : h \to \Sigma \mathrm{gl}_1 A$ of A, the composite $\Sigma(w \circ \tau) : \Sigma h \to \Sigma \mathrm{gl}_1 K(A)$ is an E_∞-twisting of $K(A)$.

Twisting iterated algebraic K-theory of *ku*

Example (K-theory)

Twisting of *ku* by 3-dimensional cohomology classes. The map $\tau : \Sigma^3 H\mathbb{Z} \to \Sigma \mathrm{gl}_1 ku$ is the delooping of the map $\mathbb{C}P^\infty \to \mathrm{GL}_1\, ku$ that regards a complex line as an invertible $\mathbb{C}$-module.

Definition (LSW)

For $\mathfrak{a}_n$ the *iterated algebraic K-theory spectrum* $\boxed{\mathfrak{a}_n := K^{(n-1)}(ku)}$, let $\tau_n : \Sigma^{n+2} H\mathbb{Z} \to \Sigma \mathrm{gl}_1 \mathfrak{a}_n$ be the *E_∞-twisting of $\mathfrak{a}_n$* obtained by an $(n-1)$-fold iteration of the procedure in Prop. * applied to $\tau = \tau_1$. This gives the twisted cohomology group $\boxed{\mathfrak{a}_n^*(X; H)}$.

Corresponding infinite loop maps $\tau_n : K(\mathbb{Z}, n+2) \to B\,\mathrm{GL}_1\, \mathfrak{a}_n$.

Example

When $n = 1$, this gives connective complex K-theory twisted by a gerbe (or, its Dixmier-Douady class in H^3): $\boxed{ku(X; H_3)}$.

Bott elements

Form periodic version:

Definition (LSW)

When n is odd, define $\boxed{\mathfrak{A}_n := \mathfrak{a}_n[\beta_n^{-1}]}$*, with* $\beta_n := (\tau_n \circ \iota) - 1 \in \pi_{n+1}\mathfrak{a}_n$*, where* ι *is the fundamental class of* $\Sigma^{n+2} H\mathbb{Z}$*.*

Definition

The twisted cohomology group $\boxed{\mathfrak{A}_n^*(X; H)}$ is the E_∞-twistings of the cohomology theory $\mathfrak{A}_n$ by $(n+2)$-dimensional cohomology classes via τ_n.

Example

- Notice that the element $\beta = \beta_1 \in \pi_2 \mathfrak{a}_1 = \pi_2 ku$, given as the composite $S^2 \to \mathbb{C}P^\infty \to \mathbb{Z} \times BU$ of the fundamental class of $\mathbb{C}P^\infty$ and its inclusion as $BU(1) \times \{1\}$, is the Bott class. Localizing ku at this class yields periodic complex K-theory: $KU := ku[\beta^{-1}]$.
- Combining, we get twisted periodic complex K-theory.

The Chern-Dold character

The Chern character has a generalization to any generalized cohomology theory. Let $\mathfrak{a_n}^*(\mathrm{pt}) = \pi_*(\mathfrak{a}_n) = R^* = \sum_j R^j$.

Definition

The *Chern-Dold character for the theory* $\mathfrak{a}_n$ is the map of cohomology theories

$$\mathrm{ch}_{\mathfrak{a}_n} : \mathfrak{a}_n^* \to \mathcal{H}^*(-; \mathfrak{a}_n^*(\mathrm{pt}) \otimes \mathbb{Q}) \,,$$

where $\mathcal{H}^i(-; \mathfrak{a}_n^*(\mathrm{pt}) \otimes \mathbb{Q}) = \sum_{m \geq 0} H^m(-; R^{i-m} \otimes \mathbb{Q})$.

- It is induced by the rationalization map

$$\mathrm{ch}_{\mathfrak{a}_n} : \mathfrak{a}_n \to (\mathfrak{a}_n)_{\mathbb{Q}} \simeq H(R_* \otimes \mathbb{Q});$$

  the rationalization of $\mathfrak{a}_n$ may be identified with the generalized Eilenberg-MacLane spectrum associated to the graded ring $R_* \otimes \mathbb{Q}$.
- Unfortunately, we have little understanding of an explicit formula for these coefficients. Further, the results of Ausoni suggest that R^* is a very complicated ring.

A higher Chern character

There is a stand-in for the Chern-Dold character with target a recognizable generalized Eilenberg-MacLane spectrum.

The map $\mathrm{ch}_n^{\geq 0} \colon \mathfrak{a}_n \to H\mathbb{Q}[\beta_n]$ can be constructed (via the Dennis trace to topological Hochschild cohomology and rationalization) and has target the generalized rational Eilenberg-MacLane ring spectrum

$$(\Sigma^\infty K(\mathbb{Z}, n+1)_+)_\mathbb{Q} \simeq H\mathbb{Q}[\beta_n]$$

whose homotopy is the ring $\mathbb{Q}[\beta_n]$.

Definition

For n odd, define the *higher Chern character* $\mathrm{ch}_n : \mathfrak{A}_n \to H\mathbb{Q}[\beta_n^{\pm 1}]$ as the localization (at β_n) of the map $\mathrm{ch}_n^{\geq 0}$.

Example

When $n = 1$, this is the usual Chern character, alternatively described as the map $KU \to KU_\mathbb{Q} \simeq (\Sigma^\infty \mathbb{C}P_+^\infty[\beta^{\pm 1}])_\mathbb{Q}$, induced by rationalization and Snaith's theorem.

Twisting the higher Chern character

Being a composition of E_∞ maps:

Proposition

The higher Chern character $\mathrm{ch}_n : \mathfrak{A}_n \to H\mathbb{Q}[\beta_n^{\pm 1}]$ is a map of E_∞ ring spectra.

So ch_n induces a map of spectra $\mathrm{ch}_n : \mathrm{gl}_1(\mathfrak{A}_n) \to \mathrm{gl}_1(H\mathbb{Q}[\beta_n^{\pm 1}])$. Further, since the composite

$$\Sigma^\infty K(\mathbb{Z}, n+1)_+ \xrightarrow{t_n} \mathfrak{a}_n \xrightarrow{\mathrm{ch}_n^{\geq 0}} (\Sigma^\infty K(\mathbb{Z}, n+1)_+)_{\mathbb{Q}}$$

is the rationalization map, then ch_n carries the twisting t_n of $\mathfrak{A}_n$ by H^{n+2} to the standard twisting of periodic cohomology by H^{n+2}.

Corollary

For each $H \in H^{n+2}(X)$, the twisted higher Chern character ch_n is a natural transformation of twisted cohomology theories:

$$\mathrm{ch}_n : \mathfrak{A}_n^*(X; H) \to H\mathbb{Q}^*(X; H)[\beta_n^{\pm 1}] \, .$$

III. Differential refinements

Algebra	Homological algebra	Higher algebra
Abelian group	Chain complex	Spectrum
Ring	DG-ring	Ring spectrum
Module	DG-module	Module spectrum

Why stacks? (in a nutshell)

G a Lie group $\rightsquigarrow$ Classifying space BG is a topological space

$$\boxed{[X, BG]} \simeq \boxed{\text{equivalence classes of } G\text{-principal bundles on } X}$$

- Shortcoming: BG does not know about:
  1. the smooth gauge transformations: G-valued functions,
  2. actual gauge fields: connections on G-principal bundles.
- Remedy: There is a smooth groupoid/smooth stack $\mathbf{B}G$:

$$\boxed{\text{maps of smooth stacks } X \to \mathbf{B}G} \simeq \boxed{G\text{-bundles on } X},$$

$$\{\text{homotopies of such maps}\} \simeq \{\text{ smooth gauge transformations}\}.$$

- *Differential refinement* to a richer smooth stack $\mathbf{B}G_\nabla$:

$$\boxed{\text{maps } X \to \mathbf{B}G_\nabla} \simeq \boxed{G\text{-Yang-Mills gauge fields on } X},$$

- True configuration space: smooth mapping stack $[X, \mathbf{B}G_\nabla]$:
  - elements are gauge fields on X,
  - morphisms are gauge transformations.

Higher $U(1)$-bundles

- Dold-Kan correspondence: Equivalence $\mathrm{Ch}^+ \simeq s\mathrm{Ab}$ between the category of (nonnegatively graded) chain complexes and the category of simplicial abelian groups.
  - The nth homology group of a chain complex is the nth homotopy group of the corresponding simplicial abelian group.
  - A chain homotopy corresponds to a simplicial homotopy.

Example

Let $\mathcal{C}$ be a chain complex that has an abelian group A in degree n and zero in other degrees. Then the corresponding simplicial group is the Eilenberg-MacLane space $K(A, n)$.

- For a small category C, its category of *presheaves* is the functor category $\mathrm{PSh}(C) := [C^{\mathrm{op}}, Set]$.
- The *sheafification functor* universally turns presheaves on C (a site) into sheaves. It is characterized as being the left adjoint functor $L : \mathrm{PSh}(C) \to \mathrm{Sh}(C)$ to the inclusion $\mathrm{Sh}(C) \hookrightarrow \mathrm{PSh}(C)$ of sheaves into all presheaves.

Machine: $\boxed{\text{Chain complexes}} \xrightarrow[\text{+stackification}]{\text{Dold-Kan}} \boxed{\text{Stacks}}$

Definition (Fiorenza-Schreiber-Stasheff)

1. The *n-stack of $U(1)$-n-bundles* (without connection) $\mathbf{B}^n U(1)$ is obtained via the "Dold-Kan correspondence" and "stackification" from the sheaf of chain complexes
$$\underline{U}(1)[n] = \left(\underline{U}(1) \to 0 \to \cdots \to 0 \right) ,$$
with $C^\infty(-; U(1))$ in degree n.

2. The *n-stack of $U(1)$-n-bundles with connections* $\boxed{\mathbf{B}^n U(1)_\nabla}$ is obtained by stackifying + Dold-Kan to the $(n+1)$-term Deligne complex
$$\underline{U}(1)[n]^\infty_D = \left(\underline{U}(1) \xrightarrow{\frac{1}{2\pi i} d\log} \Omega^1(-; \mathbb{R}) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(-; \mathbb{R}) \right) ,$$
where $\underline{U}(1)$ is the sheaf of smooth functions with values in $U(1)$, and with $\Omega^n(-; \mathbb{R})$ in degree zero.

- Equivalence classes of $U(1)$-n-bundles on X are in natural bijection with
$$H^{n+1}(X; \mathbb{Z}) \cong H^n(X; \underline{U}(1)) \cong \mathbb{H}^0(X; \underline{U}(1)[n]) \cong \pi_0 \mathbf{H}(X; \mathbf{B}^n U(1)) .$$
- Equivalence classes of $U(1)$-n-bundles with connection on smooth manifold X
$$\hat{H}^{n+1}(X; \mathbb{Z}) \cong \mathbb{H}^0(X; \underline{U}(1)[n]^\infty_D) \cong \pi_0 \mathbf{H}(X; \mathbf{B}^n U(1)_\nabla) .$$

$\mathbb{H}$ is hypercohomology of X, and $\mathbf{H}$ is the groupoid of principal n-bundles.

Obvious morphism of chain complexes of sheaves $\underline{U}(\mathbf{1})[n]^\infty_D \to \underline{U}(\mathbf{1})[n]$ induces:

Geometric realizations and smooth refinement

While smooth higher stacks have richer structure than topological spaces, there is a map called *geometric realization* that sends any smooth higher stack to the topological spaces which is the "best approximation" to it, in a precise sense. This is a "functor"

$$|-| : \mathrm{SmoothGrpd} \to \mathrm{Top}\,.$$

Examples

1. The geometric realization of the n-stack $\mathbf{B}^n\mathrm{U}(1)$ is the Eilenberg-MacLane space $K(\mathbb{Z}, n+1)$ (notice the degree shift) which classifies integral cohomology $|\mathbf{B}^n\mathrm{U}(1)| \simeq K(\mathbb{Z}, n+1)$.
2. The geometric realization of the moduli stack $\mathbf{B}\mathrm{Spin}$ of Spin-principal bundles is the ordinary classifying space $B\mathrm{Spin}$: $|\mathbf{B}\mathrm{Spin}| \simeq B\mathrm{Spin}$ (all up to weak homotopy equivalence).

Extensions and refinements

The spectral sequences can be extended to the differential refinements, that is we can discuss theory E by adjoining geometric data to it.

Theorem (Grady-S.)

We have the differential refinement of the following:

1. *Primary cohomology operations: Steenrod.*
2. *Secondary cohomology operations: Massey.*
3. *AHSS with a concrete identification of the differentials .*

$$\begin{array}{ccc}
\boxed{\text{Untwisted theory } \mathrm{E}} & \xrightarrow{\text{twist}} & \boxed{\text{Twisted theory } \mathrm{E}_{\mathrm{tw}}} \\
\Big\downarrow \textit{refinement} & & \Big\downarrow \textit{refinement} \\
\boxed{\text{Differential theory } \widehat{E}} & \xrightarrow{\text{twist}} & \boxed{\text{Twisted differential theory } \widehat{E}_{\mathrm{tw}}}
\end{array}$$

$$\begin{array}{ccc}
d & \xrightarrow{\text{twist}} & d_{\mathrm{twist}} = d + \textit{secondary operation} \\
\Big\downarrow \textit{refinement} & & \Big\downarrow \textit{refinement} \\
d_{\mathrm{refined}} = \widehat{d} & \xrightarrow{\text{twist}} & \widehat{d}_{\mathrm{twist}} = \widehat{d} + \widehat{\textit{secondary operation}}
\end{array}$$

IV(a). Computations

A twisted Atiyah-Hirzebruch spectral sequence

The cellular filtration of a CW complex gives rise to the AHSS for Morava K-theory. The same holds in the twisted case (generalizing [Atiyah-Segal]) :

Theorem (SW)

For $H \in H^{n+2}(X)$, there is a spectral sequence converging to $K(n)^*(X; H)$ with $E_2^{p,q} = H^p(X, K(n)^q)$. The first possible nontrivial differential is $d_{2^{n+1}-1}$; this is given by

$$d_{2^{n+1}-1}(xv_n^k) = (Q_n(x) + (-1)^{|x|} x \cup (Q_{n-1} \cdots Q_1(H)))v_n^{k-1}.$$

- Here Q_n is the n^{th} *Milnor primitive* cohomology operation at prime 2.
- It may be defined inductively as $Q_0 = Sq^1$, the Bockstein operation, and $Q_{j+1} = Sq^{2^j} Q_j - Q_j Sq^{2^j}$, where $Sq^j : H^n(X; \mathbb{F}_2) \to H^{n+j}(X; \mathbb{F}_2)$ is the j-th Steenrod square.

Homology vs. cohomology

- Twisted homology is generally easier to define.
- **Poincaré duality:**
  - M a compact smooth manifold with tangent bundle TM of rank d.
  - Embed M into $\mathbb{R}^N$ and do Pontrjagin-Thom construction.
  - Milnor-Spanier-Atiyah duality gives isomorphism

$$E_*(M) \cong E^*(M^{-TM}) \; .$$

  - In the presence of a Thom isomorphism

$$E^*(M^{-TM}) \cong E^{d-*}(M)$$

  this leads to Poincaré duality [Ando-Blumberg-Gepner]:

$$E_*(M) \cong E^{d-*}(M) \; .$$

Example

For M oriented, this gives for the case of K-theory:

$$K_*(M) \cong K^{d-*}(M)_{-W_3(M)} \; .$$

Algebraic structures on Morava

- $K(n)_*(X)$ is always a coalgebra.
- When X is an H-space, it is in addition a Hopf algebra.
- This will be the case for all the examples for which we compute the Morava K-theory.
- The coefficients of the theory $K(n)_* = \mathbb{Z}/p[v_n, v_n^{-1}]$ form a graded field.
- So $K(n)_*(-)$ always has a Künneth isomorphism for products

$$K(n)_*(X \times Y) \cong K(n)_*(X) \otimes_{K(n)_*} K(n)_*(Y) \, .$$

- Fibrations? No general formula but when the fiber is an Eilenberg-MacLane space: The universal coefficient theorem

Theorem ((UCT) S.-Westerland)

$K(\mathbb{Z}, n+2)$-twisted Morava K-theory $K(n)$ of a space is isomorphic to the untwisted Morava K-theory of a certain $K(\mathbb{Z}, n+1)$-bundle over that space.

This is a generalization of a theorem by Khorami in the case of twisted K-theory, i.e. for $n = 1$, where the fibers are $K(\mathbb{Z}, 2) \simeq \mathbb{C}P^\infty$.

Connected covers [S.-Schreiber-Stasheff, S.]

$$
\begin{array}{ccccc}
 & & \vdots & & \\
 & & \downarrow & & \\
K(\mathbb{Z},11) & \longrightarrow & BO\langle 13\rangle = \mathrm{BNinebrane} & & \\
 & & \downarrow & & \\
K(\mathbb{Z}/2,9) & \longrightarrow & BO\langle 11\rangle = \mathrm{B2\text{-}Spin} & \xrightarrow{\frac{1}{240}p_3} & K(\mathbb{Z},12) \\
 & & \downarrow & & \\
K(\mathbb{Z}/2,8) & \longrightarrow & BO\langle 10\rangle = \mathrm{B2\text{-}Orient} & \xrightarrow{\alpha_{10}} & K(\mathbb{Z}/2,10) \\
 & & \downarrow & & \\
K(\mathbb{Z},7) & \longrightarrow & BO\langle 9\rangle = \mathrm{BFivebrane} & \xrightarrow{\alpha_{9}} & K(\mathbb{Z}/2,9) \\
 & & \downarrow & & \\
K(\mathbb{Z},4) & \longrightarrow & BO\langle 8\rangle = \mathrm{BString} & \xrightarrow{\frac{1}{6}p_2} & K(\mathbb{Z},8) \\
 & & \downarrow & & \\
K(\mathbb{Z}/2,1) & \longrightarrow & BO\langle 4\rangle = \mathrm{BSpin} & \xrightarrow{\frac{1}{2}p_1} & K(\mathbb{Z},3) \\
 & & \downarrow & & \\
 & & BO\langle 2\rangle = \mathrm{BSO} & \xrightarrow{w_2} & K(\mathbb{Z}/2,2) \\
 & & \downarrow & & \\
 & & BO & \xrightarrow{w_1} & K(\mathbb{Z}/2,2)
\end{array}
\tag{7}
$$

Morava K-theory of classifying spaces [S.-Yarosh]

- Highlight: [Hopkins-Kuhn-Ravenel] conjecture on evenness of Morava K-theory of BG (G finite group).
- Its counterexamples and characterization of evenness [Kriz].
- Note that all Eilenberg-MacLane spaces we use have even Morava K-theory (by [Ravenel-Wilson-Yagita]).
- **Which chromatic level for Morava K-theory to use?**

  We highlight three effects:

  1. Acyclicity
  2. Vanishing
  3. Untwisting

1. Acyclicity

- **Ravenel-Wilson [RW]:** nth Morava K-theory 'sees' only the first n Eilenberg-MacLane spaces.

$$\widetilde{K}(n)_* K(G, n+2) = 0 \;, \qquad \widetilde{K}(n)_* K(\mathbb{Z}/p, n+1) = 0 \;.$$

- So stabilization: $K(n)$-(co)homology would be the same for $X\langle m\rangle$ and $X\langle m+1\rangle$ after some critical m.

Example

For String, BString, Fivebrane, BFivebrane, we need at least $K(2)$, $K(3)$, $K(6)$ and $K(7)$, respectively.

- $n = 1$: $K(1)_*(K(\mathbb{Z}, j > 2)$ is trivial. So

Lemma (Twisted $K(1)$-homology of connected covers)

(i) *All connected covers of Spin, i.e. String, Fivebrane etc. have the same twisted $K(1)$-homology.*

(ii) *All classifying spaces covering* BSpin, *i.e.* BString, BFivebrane, *etc., have the same twisted $K(1)$-homology.*

- Similarly, for higher chromatic levels: BString for $n < 2$, Fivebrane for $n < 6$, and BFivebrane for $n < 7$.

- cf. [S-Wheeler]: $\otimes\mathbb{Q}$, all covers can be described using Spin
- So analog of rationalization: Morava $K(n)$-homology only sees EM spaces of certain degrees, and the ones not seen can be viewed as rationalized as far as $K(n)$-homology goes.
- Indeed, for a fibration $F \to E \xrightarrow{j} B$ with fiber F which is $K(n)_*$-acyclic, the map j is a $K(n)_*$-equivalence (see Hopkins-RW).

RW : $K(\mathbb{Z}, i)$ is $K(n)_*$-acyclic if and only of $i > n+1$. So

Lemma ($K(n)$-equivalences for connected covers)

(i) *$K(\mathbb{Z}, 3)$ is $K(1)_*$-acyclic so that $B\mathrm{Spin}$ and $B\mathrm{String}$ are $K(1)_*$-equivalent.*

(ii) *$K(\mathbb{Z}, 6)$ is $K(n)_*$-acyclic for $n < 5$ so that String and Fivebrane are $K(n < 5)_*$-equivalent.*

(iii) *$K(\mathbb{Z}, 7)$ is $K(n)_*$-acyclic for $n < 6$ so that $B\mathrm{String}$ and $B\mathrm{Fivebrane}$ are $K(n < 6)_*$-equivalent.*

- Analogous statements for 2-Orient, 2Spin and Ninebrane, etc.
- **Compatible with Bousfield's:** For any space X, each $K(n)_*$-equivalence of spaces is a $K(m)_*$-equivalence for $1 \leq m \leq n$.

2. Vanishing of twisted Morava K-theory of certain spaces

Theorem (Vanishing theorem for twisted Morava K-theory)

If a principal $K(\mathbb{Z}, n+1)$ bundle $\xi : E \to B$ is such that the induced map on Morava homology is a map of Hopf algebras, [a] *and composition with the Bockstein map gives an exact sequence*

$$K(n)_*\left(K(\mathbb{Z}/2, n)\right) \xrightarrow{\delta_*} K(n)_*(K(\mathbb{Z}, n+1)) \longrightarrow K(n)_*(E)$$

then $K(n)_*(B, \xi) = 0$.

[a] This is, for example, the case when $E \to B$ is a loop space map.

Example (Twisted Morava K-homology of BSpin)

We have

$$K(2)_*\left(\mathrm{BSpin}; \tfrac{1}{2}p_1\right) = 0 \,,$$

where $\frac{1}{2}p_1 \in H^4(\mathrm{BSpin}; \mathbb{Z})$ is the first fractional Pontrjagin class, classifying the fibration $K(\mathbb{Z}, 3) \to \mathrm{BString} \to \mathrm{BSpin}$.

- Similar results for n-spheres and n-EM spaces generalize [Khorami] on twisted K-homology of S^3 and $K(\mathbb{Z}, 3)$

Theorem (Untwisting in the Whitehead tower)

The twisted Morava K-homology of all groups in the Whitehead tower of the orthogonal and unitary groups, and their classifying spaces, with the canonical twist, is isomorphic to the underlying untwisted Morava K-homology.

- This generalizes to higher chromatic level results by [Douglas] + (physicists) on twisted K-homology of Lie groups
- Generalizes to the twisted case results of [Kitchloo-Laures-Wilson].

Example (Twisted Morava K-homology of String and BString)

$$K(5)_* \left(\mathrm{String}; H_7\right) \cong K(5)_*(\mathrm{String}) ,$$
$$K(6)_* \left(\mathrm{BString}; \tfrac{1}{6}p_2\right) \cong K(6)_* \left(\mathrm{BString}\right) ,$$

where $\frac{1}{6}p_2$ is the second fractional Pontrjagin class, classifying the fibration $K(\mathbb{Z}, 7) \to \mathrm{BFivebrane} \to \mathrm{BString}$ and H_7 is its looping.

IV(b). Applications

Application: T-duality [Lind-S.-Westerland]

- Given a base space X, E and $\hat{E}$ are principal S^1-bundles over X, and H and $\hat{H}$ are cohomology classes in $H^3(E)$ and $H^3(\hat{E})$, respectively.
- Criterion for two sets of data (E, H) and $(\hat{E}, \hat{H})$ to be *T-dual*:
    1. Bouwknegt-Evslin-Mathai: Relations among various characteristic classes.
    2. Bunke-Schick: existence of a Thom class on a certain S^3-bundle over X into which both E and $\hat{E}$ embed.
- When (E, H) and $(\hat{E}, \hat{H})$ are a T-dual pair, the T-duality isomorphism

$$\boxed{\mathrm{K}^*(E; H) \cong K^{*-1}(\hat{E}, \hat{H})}$$

- Cohomological formulae:

$$\boxed{\pi_*(H) = c_1(\hat{E}) \ , \qquad \pi_*(\hat{H}) = c_1(E)}$$

We give a similar Thom class criterion for (E, H) and $(\hat{E}, \hat{H})$ to be T-dual in the higher (dimensional, categorical, and (conjecturally) chromatic) context, where E and $\hat{E}$ are fiber bundles over X with fiber the q-sphere S^q, and H and $\hat{H}$ are classes in $H^{2q+1}(E)$ and $H^{2q+1}(\hat{E})$, respectively.

T-dual sphere bundles

Let X be a topological space, and $\pi : E \to X$ and $\hat{\pi} : \hat{E} \to X$ be S^q-bundles over X. Define $E *_X \hat{E}$ to be the *fiberwise join* of E and $\hat{E}$ over X, i.e. fiber the join $S^q * S^q = S^{2q+1}$, and there are natural fiberwise embeddings

$$i : E \hookrightarrow E *_X \hat{E} \quad \text{and} \quad \hat{i} : \hat{E} \hookrightarrow E *_X \hat{E}$$

given by inclusion of each factor in the join.

Definition

For a S^m-bundle $p : Y \to X$, a *bundle Thom class* is an element $\mathrm{Th} \in H^m(Y;\mathbb{Z})$ with the property that its restriction to each fiber is a generator of $H^m(S^m;\mathbb{Z})$. Equivalently, $p_!(\mathrm{Th}) = 1 \in H^0(X;\mathbb{Z})$.

We let $n = 2q - 1$ $H \in H^{n+2}(E;\mathbb{Z})$ and $\hat{H} \in H^{n+2}(\hat{E};\mathbb{Z})$.

Definition

The pairs (E, H) and $(\hat{E}, \hat{H})$ are *T-dual* if there exists a bundle Thom class $\mathrm{Th} \in H^{n+2}(E *_X \hat{E})$ with $i^* \,\mathrm{Th} = H$ and $\hat{i}^* \,\mathrm{Th} = \hat{H}$.

- Consider the correspondence space $E \times_X \hat{E}$, which is an $S^q \times S^q$-bundle over X:

$$\begin{array}{ccccc} & & E \times_X \hat{E} & & \\ & \swarrow^{p} & & \searrow^{\hat{p}} & \\ E & & & & \hat{E}\,. \\ & \searrow_{\pi} & & \swarrow_{\hat{\pi}} & \\ & & X & & \end{array}$$

- There is a tautological homotopy $h : I \times (E \times_X \hat{E}) \to E *_X \hat{E}$ from $i \circ p$ to $\hat{i} \circ \hat{p}$ that is given by the formula $h_t(e, \hat{e}) = (t, e, \hat{e})$ and recognizes the fiberwise join $E *_X \hat{E}$ as a quotient of $I \times (E \times_X \hat{E})$.

- Pulling the twisting classes back over p and $\hat{p}$ gives cohomology classes $p^* H$ and $\hat{p}^* \hat{H}$, respectively, on the correspondence space. But since $i^* \operatorname{Th} = H$ and $\hat{i}^* \operatorname{Th} = \hat{H}$, we have a homotopy

$$\Lambda = \operatorname{Th} \circ h : p^* H = p^* i^* \operatorname{Th} \to \hat{p}^* \hat{i}^* \operatorname{Th} = \hat{p}^* \hat{H}.$$

Higher T-duality

Theorem (Lind-.S-Westerland)

Let $n = 2q - 1$, and assume that (E, H) and $(\hat{E}, \hat{H})$ are a T-dual pair. Let $\tau_E \colon E \to B\,\mathrm{GL}_1\,\mathfrak{A}_n$ denote the orientation twisting determined by the vertical tangent bundle of $E \to X$. Then there is an isomorphism of twisted cohomology groups

$$T := \hat{p}_! \circ \Lambda \circ p^* : \mathfrak{A}_n^*(E; \tau_E \otimes H) \to \mathfrak{A}_n^*(\hat{E}; \hat{H})$$

given in terms of a Fourier-Mukai pull-push construction on the correspondence space $E \times_X \hat{E}$. Given an $\mathfrak{A}$-orientation of the fiber bundle $E \to X$, the T-duality isomorphism takes the form

$$T : \mathfrak{A}_n^{*+q}(E; H) \to \mathfrak{A}_n^*(\hat{E}; \hat{H}).$$

- In fact, we extend the result for a *larger class* of A_∞ spectra R than just $\mathfrak{A}_n$, namely those which may be twisted by n-gerbes.

T-duality in cohomology

- If R' is an A_∞ ring spectrum and $\psi : R \to R'$ an A_∞ map, the composite $\psi \circ \phi$ is a periodic twisting of R' by n-gerbes, and so the results of the theorem also hold for R'.
- ψ throws the T-duality isomorphism for R onto that for R'.
- When $R = \mathfrak{A}_n$, $R' = H\mathbb{Q}[\beta_n^{\pm 1}]$, and $\psi = \mathrm{ch}_n$, we have:

Corollary

1. The higher Chern character ch_n throws the T-duality isomorphism for $\mathfrak{A}_n$ onto that of periodic rational cohomology:

$$\begin{array}{ccc} \mathfrak{A}_n^*(E; \tau_E \otimes H) & \xrightarrow[\cong]{T_{\mathfrak{A}_n}} & \mathfrak{A}_n^*(\hat{E}; \hat{H}) \\ \downarrow {\scriptstyle \mathrm{ch}_n} & & \downarrow {\scriptstyle \mathrm{ch}_n} \\ H\mathbb{Q}^*(E; \tau_E \otimes H)[\beta_n^{\pm 1}] & \xrightarrow[T_{H\mathbb{Q}[\beta_n^{\pm 1}]}]{\cong} & H\mathbb{Q}^*(\hat{E}; \hat{H})[\beta_n^{\pm 1}] \end{array} .$$

2. Cohomological formula: $\boxed{\pi_*(H) = e(\hat{E}) , \qquad \pi_*(\hat{H}) = e(E)}$.

Classifying spaces for pairs and triples

Let G be a topological group equipped with a map to $\mathrm{Homeo}^+(S^q)$.
We introduce spaces $R_n(G)$ and $P_n(G)$, $n = 2q - 1$, that classify the objects considered earlier.

1. $[X, R_n(G)]$ is the set of isomorphism classes of pairs (E, H), where $E \to X$ is an S^q-bundle with structure group G, and $H \in H^{n+2}(E)$ is a twisting class.
2. $[X, P_n(G)]$ is the set of isomorphism classes of triples $(E, \hat{E}, \mathrm{Th})$, where E, $\hat{E}$ are S^q-bundles with structure group G, and $\mathrm{Th} \in H^{n+2}(E *_X \hat{E})$ is a bundle Thom class.

- There is a map $f : P_n(G) \to R_n(G)$ that induces the natural transformation which carries $(E, \hat{E}, \mathrm{Th})$ to $(E, i^* \mathrm{Th})$.

Proposition

If the Euler class map $e : BG \to K(\mathbb{Z}, q + 1)$ is m-connected, so too is the comparison map $f : P \to R$. Thus, over complexes of dim $< m$, there exists a unique T-dual $(\hat{E}, \hat{H})$ for any (E, H). In dimension m, such duals exist, but are not necessarily unique.

Example (Principal circle bundles)

When $n = q = 1$ and $G = U(1)$, the Euler class $e : BU(1) \to K(\mathbb{Z}, 2)$ is a weak homotopy equivalence; therefore the map f is as well. This recovers the result of Bunke-Schick that in the case of circle bundles and $U(1)$-gerbes, T-dual pairs exist and are unique up to equivalence.

Example (Principal SU(2) bundles)

$n = 5$, i.e. $q = 3$, and $G = SU(2)$, and consider pairs (E, H) consisting of a principal $SU(2)$-bundle $\pi\colon E \to X$ and a 5-gerbe $H\colon E \to K(\mathbb{Z}, 7)$ on E. The Euler class $e : BSU(2) \to K(\mathbb{Z}, 4)$ is 5-connected, and so the same holds for f. This recovers result of Bouwknegt-Evslin-Mathai that when the base X has dim ≤ 4, the T-dual of (E, H) exists and is unique.

Example (Non-principal bundles)

Also, by taking G to be a topological group with a map to $\mathrm{Homeo}^+(S^q)$, we can recover results, for $q = 1$, on **non-principal** circles bundles by Baraglia and Mathai-Rosenberg and for $q = 3$ on non-principal $SU(2)$ bundles by Bouwknegt-Evslin-Mathai.

Example (T-duality for principal SU(2) bundles in our theory)

- The map
$$\tau_5 \colon K(\mathbb{Z},7) \to B\,\mathrm{GL}_1\,K^{(4)}(ku)[\beta_5^{-1}] = B\,\mathrm{GL}_1\,\mathfrak{A}_5$$
defines a twisting of $\mathfrak{A}_5$ by 5-gerbes and we now consider the T-duality isomorphism.
- Since the total space of the universal principal $SU(2)$-bundle is contractible, the vertical tangent bundle of E is trivializable.
- Hence the associated twist τ_E of $\mathfrak{A}_5$ is trivial and the T-duality isomorphism takes the form
$$T = \hat{p}_! \circ \Lambda \circ p^* \colon \mathfrak{A}_5^*(E,H) \to \mathfrak{A}_5^{*-3}(\hat{E},\hat{H})\ .$$
- As we saw, the Chern character throws this T-duality isomorphism onto a T-duality isomorphism in ordinary cohomology
$$T \colon H\mathbb{Q}^*(E;H)[\beta_5^{-1}] \xrightarrow{\cong} H\mathbb{Q}^{*-3}(\hat{E};\hat{H})[\beta_5^{-1}]\ .$$
- This recovers the T-duality isomorphism for principal $SU(2)$-bundles studied by Bouwknegt-Evslin-Mathai.

Thank you!