Universal Principal Bundles And Classifying Spaces

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Abstract

The content of this document is intended to be complementary notes to the course of Pr. Abdelhak Abouqateb presented as part of the 2018 international conference "Ecole Mathématique Africaine" (EMA) at Rabat Morocco. In these few notes, we treat the theory of principal bundles from the perspective of algebraic topology.

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1. Introduction

Let X be a compact Hausdorff space. Swan's theorem [Swa62] states that any rank k vector bundle $E \xrightarrow{\pi} X$ is a subbundle of the trivial vector bundle $X \times \mathbb{R}^n$ for some $n \in \mathbb{N}$. There is another way of stating this result in the following manner : For every $p \in X$, the fiber E_p is a k-dimensional subspace of $\mathbb{R}^n$, thus we can construct a map $f : X \longrightarrow \mathrm{Gr}_{\mathbb{R}}(k,n)$ given by $f(p) = E_p$, we can check that f is continuous. Now if we define :

$$E_{k,n}(\mathbb{R}) = \{(\ell, v) \in \mathrm{Gr}_{\mathbb{R}}(k,n) \times \mathbb{R},\ v \in \ell\ \}.$$

We can check that $E_{k,n}(\mathbb{R})$ is a rank k vector bundle over $\mathrm{Gr}_{\mathbb{R}}(k,n)$, furthermore Swan's theorem is equivalent to claiming that $E \simeq f^*(E_{k,n}(\mathbb{R}))$. Now this is far from being a classification result since the integer n depends on many factors, namely the space X, the rank k and the vector bundle E (i.e $n = n(k, X, E)$), to resolve this it is sufficient to remark that when E is a vector subbundle of $X \times \mathbb{R}^n$ then it is a subbundle of $X \times \mathbb{R}^m$ for every

$m \geq n$. So we define the "infinite" Grassmannian $\mathrm{Gr}_{\mathbb{R}}(k,\infty)$ as the collection of k-dimensional subspaces of $\mathbb{R}^\infty$, and since E is a subbundle of $X \times \mathbb{R}^\infty$ we define a continuous map $f : X \longrightarrow \mathrm{Gr}_{\mathbb{R}}(k,\infty)$, $p \mapsto E_p$. Now define the rank k vector bundle $E_k(\mathbb{R})$ over $\mathrm{Gr}_{\mathbb{R}}(k,\infty)$ by the formula :

$$E_k(\mathbb{R}) = \{(\ell, v) \in \mathrm{Gr}_{\mathbb{R}}(k,\infty) \times \mathbb{R},\ v \in \ell\ \},$$

we obtain as before that $E \simeq f^*(E_k(\mathbb{R}))$, the complex vector bundle case is exactly the same. Now if $\hat{E} \longrightarrow Y$ is a vector bundle, it is well known that homotopic maps $f, g : X \longrightarrow Y$ give rise to isomorphic vector bundles, in summary we get the following statement :

For every $k \in \mathbb{N}$ there exists a rank k vector bundle $E_k(\mathbb{R}) \longrightarrow \mathrm{Gr}_{\mathbb{K}}(k,\infty)$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) satisfying the following property : For every rank k vector bundle $E \longrightarrow X$ over a compact Hausdorff space, there exists a homotopy class $f : X \longrightarrow \mathrm{Gr}_{\mathbb{K}}(k,\infty)$ such that $E \simeq f^*(E_k(\mathbb{R}))$, in other words we get that the map :

$$[X, \mathrm{Gr}_{\mathbb{K}}(k,\infty)] \longrightarrow \mathrm{Vect}_k(X,\mathbb{R}), \quad [f] \mapsto [f^*(E_k(\mathbb{R}))]$$

is surjective. We can show in fact that this is a bijective correspondance.

Since there is a bijective correspondance between vector bundles of rank k with structure group $G \subset \mathrm{GL}(k,\mathbb{R})$ and principal G-bundles, the preceding statement can be reformulated as follows :

For every matrix group G, there exists a principal G-bundle $EG \longrightarrow BG$ satisfying the following property : For every principal G-bundle $P \longrightarrow X$ over a compact Hausdorff space, there exists a unique homotopy class $f : X \longrightarrow BG$ such that $P \simeq f^*(EG)$, in other words we get that the map :

$$[X, BG] \longrightarrow \mathrm{Prin}_G(X), \quad [f] \mapsto [f^*(EG)]$$

is a bijective correspondance.

Now the natural question to ask is whether this classification theorem extends to the case when G is any topological group. The answer is affirmative for CW-complexes and the rest of this course is about proving this claim, the proof will not rely on vector bundles, in fact the classification theorem for vector bundles will be obtained as a consequence which will be discussed in the final section.

2. Homotopy properties of fiber bundles

We begin by a quick review of the homotopy invariance of the pullback operation, the most important result of the section is the following theorem due to *Steenrod* :

Theorem 2.0.1 (Covering Homotopy theorem)**.** *Let $E \xrightarrow{\pi} B$ and $\hat{E} \xrightarrow{\hat{\pi}} \hat{B}$ two fiber bundles with the same fiber type F and consider a bundle map $\tilde{f} : E \longrightarrow \hat{E}$ over some map $f : B \longrightarrow \hat{B}$ (i.e $\hat{\pi} \circ \tilde{f} = f \circ \pi$). Suppose that $H : B \times I \longrightarrow \hat{B}$ is a homotopy such that $H_0 = f$, then there exists a homotopy $\tilde{H} : E \times I \longrightarrow \hat{E}$ whose induced homotopy is H, i.e :*

$$\begin{array}{ccc} E \times I & \xrightarrow{\tilde{H}} & \hat{E} \\ {\scriptstyle (\pi, \mathrm{Id})}\downarrow & & \downarrow{\scriptstyle \hat{\pi}} \\ B \times I & \xrightarrow[H]{} & \hat{B} \end{array} \tag{1}$$

Moreover $\tilde{H}_0 = \tilde{f}$.

The rest is just a consequence of the Covering homotopy theorem :

Theorem 2.0.2. *Let $E \xrightarrow{\pi} B$ be a fiber bundle with fiber type F and suppose that $f_0 : X \longrightarrow B$ and $f_1 : X \longrightarrow B$ are homotopic maps. Then the pullback bundles are isomorphic, i.e $f_0^*(E) \simeq f_1^*(E)$.*

Proof. Define a homotopy $H : X \times I \longrightarrow B$ between f_0 and f_1. Since the fiber bundles $f_0^*(E) \xrightarrow{pr_2} X$ and $E \xrightarrow{\pi} B$ have the same fiber type, the covering homotopy theorem gives that there exists a map $\tilde{H} : f_0^*(E) \times I \longrightarrow E$ such that $\pi \circ \tilde{H} = H \circ pr_2$, from the universal property of the pullback bundle, $\tilde{H}$ can be seen as a fiber bundle homomorphism :

$$\begin{array}{ccc} f_0^*(E) \times I & \xrightarrow{\tilde{H}} & H^*(E) \\ \downarrow & & \downarrow \\ X \times I & \xrightarrow[\mathrm{Id}]{} & X \times I \end{array} \tag{2}$$

To check that $\tilde{H} : f_0^*(E) \times I \longrightarrow H^*(E)$ is an isomorphism, it suffices to notice that $\tilde{H}$ induce the identity when restricted to the fibers. To conclude, we remark that $\tilde{H}(f_0^*(E) \times \{1\}) = f_1^*(E)$. □

It is worth to mention that when $f_0, f_1 : X \longrightarrow B$ are homotopic maps and $E \xrightarrow{\pi} B$ is a principal G-bundle and then the pullbacks $f_0^*(E)$ and $f_1^*(E)$ are isomorphic *as principal G-bundles*. Similarly, if E is a vector bundle we get that the pullbacks $f_0^*(E)$ and $f_1^*(E)$ are isomorphic *as vector bundles*.

Corollary 2.0.1. *Let $E \xrightarrow{\pi} B$ be a fiber bundle and suppose that B is a contractible space. Then E is trivial.*

3. CW-complexes

We begin by recalling the notion of an adjunction space. Let X, Y be two topological spaces, $A \subset X$ a closed subset and $f : A \longrightarrow Y$ a continuous map. Define on the disjoint union $X \sqcup Y$ the equivalence relation : for all $x \in X$ and $y \in Y$,

$$x \sim y \quad \text{if and only if} \quad x \in A \text{ and } y = f(x).$$

The adjunction space of X and Y (via f) is the topological space $X \sqcup Y/\sim$ endowed with the quotient topology which we denote by $X \cup_f Y$. Denote $p : X \sqcup Y \longrightarrow X \cup_f Y$ the canonical projection.

Proposition 3.0.1. *We have the following properties :*

1. *The canonical projection induce an imbedding $p_{|Y} : Y \longrightarrow X \cup_f Y$ of Y onto a closed subspace and an imbedding $p_{|X \setminus A} : X \setminus A \longrightarrow X \cup_f Y$ of $X \setminus A$ onto an open subspace.*
2. *The adjunction space $X \cup_f Y$ is a Hausdorff space whenever X and Y are Hausdorff spaces.*
3. *If X and Y are normal spaces, then $X \cup_f Y$ is a normal space.*

We turn now to the notion of CW-complex. Basically, given a family of topological spaces $X^0 \hookrightarrow X^1 \hookrightarrow \ldots \hookrightarrow X^n \hookrightarrow \ldots$, the colimit of such a family is the set $X = \cup_{n \in \mathbb{N}} X^n$ endowed with the weak topology : A subset $U \subset X$ is open if and only if $U \cap X^n$ is open in X^n for all $n \in \mathbb{N}$. We write $X = \underset{n \to +\infty}{\operatorname{colim}} X^n$. Now a CW-complex is just a colimit space $X = \underset{n \to +\infty}{\operatorname{colim}} X^n$ where the spaces X^n are defined inductively in the following manner :

1. We start with a discrete space X^0.
2. Suppose that X^{n-1} is defined, then choose a family of attaching maps $\varphi_\alpha : \partial D^n_\alpha \subset D^n_\alpha \longrightarrow X^{n-1}$ and put $X^n := (\sqcup_\alpha D^n_\alpha) \underset{\varphi_\alpha,\, \alpha}{\cup} X^{n-1}$.

For all $n \in \mathbb{N}$, let $p_n : (\sqcup_\alpha D^n_\alpha) \sqcup X^{n-1} \longrightarrow X^n$ be the canonical projection. An n-dimensional cell in X is the embedded image $e^n_\alpha = (\iota_n \circ p_n)(D^n_\alpha \setminus \partial D^n_\alpha)$ where $\iota_n : X^n \longrightarrow X$ is the inclusion of X^n in X. To each cell e^n_α we can associate a characteristic map $\phi^n_\alpha : D^n_\alpha \longrightarrow X$ which is by definition the composition :

$$D^n_\alpha \overset{inc}{\hookrightarrow} (\sqcup_\beta D^n_\beta) \sqcup X^{n-1} \overset{p_n}{\longrightarrow} X^n \overset{\iota_n}{\hookrightarrow} X.$$

This is clearly a continuous map, and it is straightforward to check that it defines a homeomorphism $\mathrm{int}(D^n_\alpha) \longrightarrow e^n_\alpha$ and that $\phi^n_{\alpha|\partial D^n_\alpha} = \iota_{n-1} \circ \varphi^n_\alpha$.

Proposition 3.0.2. *A set $A \subset X$ is open (resp. closed) if and only if the set $(\phi^n_\alpha)^{-1}(A)$ is open (resp. closed) in D^n_α for every characteristic map ϕ^n_α.*

Proof. The continuity of the characteristic maps ϕ^n_α assures that $(\phi^n_\alpha)^{-1}(A)$ is open for every open set $A \subset X$. Conversely, let $A \subset X$ and assume that $(\phi^n_\alpha)^{-1}(A)$ is open in D^n_α for every characteristic map ϕ^n_α, we will show inductively that $A \cap X^k$ is open in X^k for all $k \in \mathbb{N}$.
For $k = 0$, $A \cap X^0$ is immediately open in X^0 since X^0 is a discrete space. Now let $k \in \mathbb{N}^*$ and suppose that $A \cap X^{k-1}$ is open in X^{k-1}, now denote $p_k : (\sqcup_\alpha D^k_\alpha) \sqcup X^{k-1} \longrightarrow X^k$ the canonical projection, then :

$$p_k^{-1}(A) = (\sqcup_\alpha (\phi^k_\alpha)^{-1}(A)) \sqcup (A \cap X^{k-1})$$

which is an open set of $(\sqcup_\alpha D^k_\alpha) \sqcup X^{k-1}$ by hypothesis. Thus $A \cap X^k$ is open in X^k. We conclude that A is open in X. □

Many important spaces admit a CW-complex structure, we state some of the in the following theorem :

Theorem 3.0.1. *Any smooth manifold admits a CW-complex structure. Similarly, any closed topological manifold of dimension $\neq 4$ admits a CW-complex structure.*

In general, an arbitrary topological space may not admit a CW-complex structure and might not even be homotopy equivalent to a CW-complex. However there is still a way of "approaching" topological spaces of interest from the point of view of homotopy theory which we now present :

Given a map of topological spaces $f : X \longrightarrow Y$, we say that f is a *weak homotopy equivalence* if the induced morphism $f^* : \pi_n(Y) \longrightarrow \pi_n(X)$ is an isomorphism (clearly any homotopy equivalence is a weak homotopy equivalence). Now a *CW-approximation* of a space Y is just a weak homotopy equivalence $f : X \longrightarrow Y$ such that X is a CW-complex. The existence of CW-approximations is guaranteed by the following theorem :

Theorem 3.0.2. *Any topological space admits a CW-approximation.*

We call a subcomplex of a CW-complex X any subspace $A \subset X$ which is a unions of cells of X such that the closure of each cell $e^n_\alpha \subset A$ is contained in A. This induce a CW-complex structure on A itself. The obvious example of a subcomplex of X are the n-skeletons X^n, to see this notice that X^n is closed in X thus given a k-dimensional cell e^k_α of X with $k \leq n$, its closure must be in X^n. Since X^n is the union of such cells, it is a subcomplex of X. (In fact X^n is the maximal n-dimensional subscomplex of X).

A pair (X, A) consisting to a CW-complex X and a subscomplex A is called a CW-pair. Finally a subcomplex $A \subset X$ is said to be finite if it is the union of a finite number of cells of X.

4. Universal principal bundles and classifying spaces

We begin by recalling some properties of principal bundles. A morphism of principal G-bundles $P_1 \xrightarrow{\pi_1} B_1$ and $P_2 \xrightarrow{\pi_2} B_2$ is any G-equivariant map $\phi : P_1 \longrightarrow P_2$. Any morphism of principal G-bundles $\phi : P_1 \longrightarrow P_2$ lies over a map $\tilde{\phi} : B_1 \longrightarrow B_2$, i.e the following diagram commutes :

$$\begin{array}{ccc} P_1 & \xrightarrow{\phi} & P_2 \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_2} \\ B_1 & \xrightarrow{\tilde{\phi}} & B_2 \end{array} \qquad (3)$$

The map $\tilde{\phi} : B_1 \longrightarrow B_2$ is uniquely defined by $\tilde{\phi}(b) = \pi_2(\phi(p))$ where $b = \pi_1(p)$. It is well-defined since given $p, q \in \pi_1^{-1}(b)$ we have that $q = p \cdot g$ for some $g \in G$ and thus :

$$\pi_2(\phi(q)) = \pi_2(\phi(p \cdot g)) = \pi_2(\phi(p) \cdot g) = \pi_2(\phi(p)).$$

To check that it is continuous, choose an open neighborhood U of b in B_1 over which $P_1 \xrightarrow{\pi_1} B_1$ is trivial and a local section $s_1 : U \longrightarrow P_1$. Then we can easily check that $\tilde{\phi}_{|U} = \pi_2 \circ \phi \circ s_1$, hence $\tilde{\phi}$ is continuous on U and since U was arbitray we obtain that $\tilde{\phi}$ is continuous on B_1.

Proposition 4.0.1. *Let $\phi : P_1 \longrightarrow P_2$ be a morphism of principal G-bundles $P_1 \xrightarrow{\pi_1} B$ and $P_2 \xrightarrow{\pi_2} B$ lying over the identity $\mathrm{Id} : B \longrightarrow B$. Then ϕ is an isomorphism.*

Proof. Let $p, q \in P_1$ such that $\phi(p) = \phi(q)$. Since $\pi_2 \circ \phi = \pi_1$ we obtain that $\pi_1(p) = \pi_1(q)$ thus $q = p \cdot g$ for some $g \in G$ and consequently

$$\phi(q) = \phi(p) \cdot g = \phi(p).$$

The group G acts freely on P_2, hence $g = e_G$ and $p = q$. This shows that ϕ is injective. For surjectivity, take $p_2 \in P_2$ and put $b = \pi_2(p_2)$, now choose any $p \in \pi_1^{-1}(b)$ then we have that :

$$\pi_2(\phi(p)) = \pi_1(p) = b = \pi_2(p_2).$$

This gives that $\phi(p) = p_2 \cdot g$ for some $g \in G$. Now putting $p_1 = p \cdot g^{-1}$ we obtain that $\phi(p_1) = \phi(p) \cdot g^{-1} = p_2$ which shows that ϕ is surjective. It is only left to show that $\phi^{-1} : P_2 \longrightarrow P_1$ is continuous. For this we consider local trivializations $\varphi_1 : U \times G \longrightarrow \pi_1^{-1}(U)$ and $\varphi_2 : U \times G \longrightarrow \pi_2^{-1}(U)$. Since ϕ is surjective and $\pi_2 \circ \phi = \pi_1$, we obtain that $\phi\big(\pi_1^{-1}(U)\big) = \pi_2^{-1}(U)$. Thus the composite map :

$$\varphi_2^{-1} \circ \phi \circ \varphi_1 : U \times G \longrightarrow U \times G$$

is a well-defined bijective morphism of principal G-bundles lying over the identity of U, hence it is automatically of the form :

$$(\varphi_2^{-1} \circ \phi \circ \varphi_1)(x, g) = (x, \tau(x, g)) = (x, \tau(x, e) \cdot g),$$

where $\tau : U \times G \longrightarrow G$ is a G-equivariant map. It is now straightforward to check that the inverse map $\varphi_1^{-1} \circ \phi^{-1} \circ \varphi_2 : U \times G \longrightarrow U \times G$ is given by :

$$(\varphi_1^{-1} \circ \phi^{-1} \circ \varphi_2)(x, g) = (x, \tau(x, e)^{-1} \cdot g),$$

which is clearly continuous since $G \longrightarrow G$, $g \mapsto g^{-1}$ is continuous. We conclude that ϕ^{-1} is continuous on $\pi_2^{-1}(U)$, and since U is arbitrarily chosen, it is globally continuous. $\square$

Recall that if $P \xrightarrow{\pi} B$ is a principal G-bundle and F is a left G-space endowed with a topological action $\rho : G \longrightarrow \mathrm{Aut}(F)$, then we can define the associated fiber bundle $P \times_G F \xrightarrow{\pi_F} B$ to P by ρ where $\pi_F([p, f]) = \pi(p)$ (the space $P \times_G F$ is the quotient of $P \times F$ by the equivalence relation $(p, f) \sim (p \cdot g, g^{-1} \cdot f)$). When $P \xrightarrow{\pi} B$ is trivial, then $P \times_G F \xrightarrow{\pi_F} B$ is also trivial.

Proposition 4.0.2. *Let $P \xrightarrow{\pi} B$ be a principal G-bundle, F a left G-space and $E = P \times_G F$ the associated bundle. For any open set $U \subset B$, there is a bijective correpondance $\Gamma(U, E) \longleftrightarrow \mathrm{Map}(\pi^{-1}(U), F)^G$.*

Proof. Let $\phi : \pi^{-1}(U) \longrightarrow F$ be a G-equivariant map and define $s_\phi : U \longrightarrow E$ by the formula $s_\phi(b) = [p, \phi(p)]$. Notice that s_ϕ is well defined, since given $p, q \in \pi^{-1}(b)$ we can write $q = p \cdot g$ and thus :

$$[q, \phi(q)] = [p \cdot g, \phi(p \cdot g)] = [p \cdot g, g^{-1} \cdot \phi(p)] = [p, \phi(p)].$$

It is clear that $\pi_F \circ s_\phi = \mathrm{Id}_U$, to check that s_ϕ is continuous we consider an open set $V \subset U$ over which $P \xrightarrow{\pi} B$ is trivial and a local section $\sigma_V : V \longrightarrow P$. Hence for all $b \in V$, $s_\phi(b) = [\sigma_V(b), (\phi \circ \sigma_V)(b)]$ which shows that $s_{\phi|V}$ is continuous, since $V \subset U$ is arbitrary, s_ϕ is continuous on U. In summary, $s_\phi \in \Gamma(U, E)$ and we have constructed a map :

$$\mathrm{Map}(\pi^{-1}(U), F)^G \longrightarrow \Gamma(U, E), \quad \phi \mapsto s_\phi.$$

We show in what follows that this map is bijective by constructing its inverse : Choose $s \in \Gamma(U, E)$ and define $\phi_s : \pi^{-1}(U) \longrightarrow F$ by the formula :

$$\phi_s(p) = f \quad \text{when } s(\pi(p)) = [p, f].$$

To see that this if well-defined write $s(\pi(p)) = [p, f_1] = [p, f_2]$, then $(p, f_1) = (p \cdot g, g^{-1} \cdot f_2)$ and since the action of G on P is free we obtain that $g = e_G$ and $f_1 = f_2$. Hence we have that :

$$s(\pi(p)) = [p, \phi_s(p)].$$

Furthermore for any $g \in G$ and any $p \in \pi^{-1}(U)$ we have that $s(\pi(p \cdot g)) = [p \cdot g, \phi_s(p \cdot g)]$ and $s(\pi(p)) = [p, \phi_s(p)]$, since $\pi(p \cdot g) = \pi(p)$ we obtain that

$$[p \cdot g, \phi_s(p \cdot g)] = [p, \phi_s(p)].$$

Thus there exists $h \in G$ such that $(p \cdot g, \phi_s(p \cdot g)) = (p \cdot h, h^{-1} \cdot \phi_s(p))$, again because the action of G on P is free we obtain that $g = h$ and $\phi_s(p \cdot g) = g^{-1} \cdot \phi_s(p)$ which gives that ϕ_s is G-equivariant. Now we check that $\phi_s : \pi^{-1}(U) \longrightarrow F$ is continuous : Consider a local trivialization $\varphi : V \times G \longrightarrow \pi^{-1}(V)$ of $P \xrightarrow{\pi} B$ over $V \subset U$, this induce a local trivialization $\varphi_F : V \times F \longrightarrow E_{|V}$ given by :

$$\varphi_F(b, f) = [\varphi(b, e), f].$$

Now write $\varphi^{-1}(q) = (\pi_F(q), \tau_F(q)$ where $\tau_F : E_{|V} \longrightarrow F$ is a G-equivariant map, thus we obtain that :

$$(b, f) = \varphi_F^{-1}[\varphi(b, e), f] = ((\pi \circ \varphi)(b, e), \tau_F[\varphi(b, e), f]) = (b, \tau_F[\varphi(b, e), f]).$$

Thus for every $f \in F$ we have that $f = \tau_F[\varphi(b, e), f]$. Hence we get that :

$$\begin{aligned} (\tau_F \circ s)(b) = (\tau_f \circ s \circ pr_1)(b, e) &= (\tau_F \circ s \circ \pi \circ \varphi)(b, e) \\ &= \tau_F[\varphi(b, e), (\phi_s \circ \varphi)(b, e)] \\ &= (\phi_s \circ \varphi)(b, e). \end{aligned}$$

Thus for every $b \in V$ and $g \in G$:

$$(\phi_s \circ \varphi)(b, g) = \phi_s(\varphi(b, e) \cdot g) = g^{-1} \cdot (\phi_s \circ \varphi)(b, e) = g^{-1} \cdot (\tau_F \circ s)(b).$$

This gives that $\phi_s \circ \varphi$ is continuous on $V \times G$, hence ϕ_s is continuous on $\pi^{-1}(V)$. Since $V \subset U$ is arbitrary we conclude that ϕ_s is globally continuous. We have thus constructed a map :

$$\Gamma(U, E) \longrightarrow \mathrm{Map}(\pi^{-1}(U), F)^G, \quad s \mapsto \phi_s.$$

It is straightforward to check that it is the inverse of the first map. □

An important consequence is the following :

Corollary 4.0.1. *Given two principal G-bundles $P \xrightarrow{\pi} B$ and $P' \xrightarrow{\pi'} B'$. There is a bijective correspondance* $\mathrm{Mor}_G(P, P') \longleftrightarrow \Gamma(B, P \times_G P')$.

Proof. It is sufficient to notice that $\mathrm{Mor}(P, P') = \mathrm{Map}(P, P')^G$ and use the preceding proposition. □

We recall that if $P \xrightarrow{\pi} B$ is a principal G-bundle and $f : X \longrightarrow B$ is a map then the pullback bundle $f^*(P) \longrightarrow X$ is a principal G-bundle. Also, if $f_0, f_1 : X \longrightarrow B$ are homotopic maps, then there is a principal G-bundle isomorphism $f_0^*(P) \simeq f_1^*(P)$. Thus, if we denote $[X, B]$ the space of homotopic classes $X \longrightarrow B$, there is a well-defined map

$$[X, B] \longrightarrow \mathrm{Prin}_G(X) \qquad [f] \mapsto [f^*P].$$

In what follows, we treat the following question :

QUESTION: *For which conditions on X and $P \xrightarrow{\pi} B$ the map $[X, B] \longrightarrow \mathrm{Prin}_G(X)$ is a bijection ?*

Recall that a space F is said to be weakly-contractible when $\pi_n(F) = 0$ for all $n \in \mathbb{N}$.

Definition 4.0.1. *A principal G-bundle $EG \xrightarrow{\pi} BG$ is said to be universal if the total space EG is weakly contractible.*

We will show that when X admits a CW-complex structure and $EG \xrightarrow{\pi} BG$ is a universal principal G-bundle, the $[X, BG] \longrightarrow \mathrm{Prin}_G(X)$ is a bijection. We start with an important lemma :

Lemma 4.0.1. *Let (B, A) be a CW-pair and $E \xrightarrow{\pi} B$ a fiber bundle with fibre F. Suppose that $\pi_k(F) = 0$ whenever $B \backslash A$ contains a $(k+1)$-dimensional cell, then every section $s \in \Gamma(A, E)$ can be extended to a global section $\tilde{s} \in \Gamma(B, E)$. In particular, when $A = \emptyset$ and F is weakly contractible, the fiber bundle $E \xrightarrow{\pi} B$ admits a global section.*

Proof. We will show inductively that the section $s_{|A^k} : A^k \longrightarrow E$ can be extended to a section $\tilde{s}^k : B^k \longrightarrow E$. For $k = 0$, it is clear that any extension of $s_{|A^0}$ is automatically continuous since B^0 is a discrete space. Let $k \in \mathbb{N}$ and suppose that $s_{|A^k} : A^k \longrightarrow E$ can be extended to a section $\tilde{s}^k \in \Gamma(B^k, E)$. We denote an arbitrary $(k+1)$-dimensional cell in B by e_α^{k+1} and we put $C_1^{k+1} = \{\alpha,\ e_\alpha^{k+1} \subset A\}$ and $C_2^{k+1} = \{\alpha,\ e_\alpha^{k+1} \subset B \setminus A\}$. We distinguish two situations :

1. Either $C_2^{k+1} = \emptyset$ (there are no $(k+1)$-dimensional cells in $B \setminus A$), then we define $\hat{s}^{k+1} : (\sqcup_\alpha D_\alpha^{k+1}) \sqcup B^k \longrightarrow E$ in the following manner :

$$\hat{s}^{k+1} = \tilde{s}^k \text{ on } B^k \quad \text{and} \quad \hat{s}^{k+1} = s \circ \phi_\alpha^{k+1} \text{ on } D_\alpha^{k+1},$$

where $\phi_\alpha^{k+1} : D_\alpha^{k+1} \longrightarrow B$ is the characteristic map of e_α^{k+1} which satisfies $\phi_\alpha^{k+1}(D_\alpha^{k+1}) \subset A$. It is clear that $\hat{s}^{k+1}$ is continuous, to show that it induce a continuous map on B^{k+1} we choose an attaching map $\varphi_\alpha^{k+1} : \partial D_\alpha^{k+1} \longrightarrow B^k$ (the map φ_α^{k+1} is just ${\phi_\alpha^{k+1}}_{|\partial D_\alpha^{k+1}}$), we easily check that $\hat{s}^{k+1}(x) = \hat{s}^{k+1}(\varphi_\alpha^{k+1}(x))$ for every $x \in \partial D_\alpha^{k+1}$. Thus $\hat{s}^{k+1}$ induce a continuous map $\tilde{s}^{k+1} : B^{k+1} \longrightarrow E$. It is straightforward to check that $\pi \circ \tilde{s}^{k+1} = \mathrm{Id}$ and that $\tilde{s}^{k+1}_{|A^{k+1}} = s_{|A^{k+1}}$, thus $\tilde{s}^{k+1} : B^{k+1} \longrightarrow E$ is a section that extends $s_{|A^{k+1}}$.

2. Suppose now that $C_2^{k+1} \neq \emptyset$. Let e_α^{k+1} be a $(k+1)$-dimensional cell contained in $B \setminus A$ and denote $\phi_\alpha^{k+1} : D_\alpha^{k+1} \longrightarrow B$ its characteristic map. Since $\phi_\alpha^{k+1}(\partial D_\alpha^{k+1}) \subset B^k$, then the composition map :

$$\tilde{s}_\alpha^k := \tilde{s}^k \circ \phi_\alpha^{k+1} : \partial D_\alpha^{k+1} \longrightarrow E$$

is well-defined and satisfies $\pi \circ \tilde{s}_\alpha^k = {\phi_\alpha^{k+1}}_{|\partial D_\alpha^{k+1}}$, thus it defines a section of the pullback bundle $({\phi_\alpha^{k+1}}_{|\partial D_\alpha^{k+1}})^*(E)$ over ∂D_α^{k+1}. Now since D_α^{k+1} is a contractible space, we get that the pullback bundle $(\phi_\alpha^{k+1})^*(E) \longrightarrow D_\alpha^{k+1}$ is trivial, i.e $(\phi_\alpha^{k+1})^*(E) \simeq D_\alpha^{k+1} \times F$. Thus $({\phi_\alpha^{k+1}}_{|\partial D_\alpha^{k+1}})^*(E) \longrightarrow \partial D_\alpha^{k+1}$ is also trivial as shown in the following commutative diagram :

$$\begin{array}{ccccc} \partial D_\alpha^{k+1} \times F & \overset{\iota}{\hookrightarrow} & D_\alpha^{k+1} \times F & \overset{\text{pullback}}{\longrightarrow} & E \\ \downarrow{\scriptstyle \mathrm{pr}_1} & & \downarrow{\scriptstyle \mathrm{pr}_1} & & \downarrow \\ \partial D_\alpha^{k+1} & \overset{\iota}{\hookrightarrow} & D_\alpha^{k+1} & \overset{\phi_\alpha}{\longrightarrow} & B \end{array} \tag{4}$$

Hence $\tilde{s}_\alpha^k$ can be seen as a map $x \mapsto (x, \tilde{\tau}_\alpha^k(x))$ where $\tilde{\tau}_\alpha^k : \partial D_\alpha^{k+1} \longrightarrow F$. Since by hypothesis $\pi_k(F) = 0$, we obtain that $\tilde{\tau}_\alpha^k$ can be extended to a continuous map $\hat{\tau}_\alpha^{k+1} : D_\alpha^{k+1} \longrightarrow F$ and thus $\tilde{s}_\alpha^k$ can be extended to a continuous section $\hat{s}_\alpha^{k+1} : D_\alpha^{k+1} \longrightarrow (\phi_\alpha^{k+1})^*(E)$. Finally we define $\hat{s}^{k+1} : (\sqcup_\alpha \alpha D_\alpha^{k+1}) \sqcup B^k \longrightarrow E$ in the following manner :

$$\hat{s}^{k+1} = \begin{cases} s \circ \phi_\alpha^{k+1} & \text{on } D_\alpha^{k+1} \text{ for } \alpha \in C_1^{k+1} \\ \hat{s}_\alpha^{k+1} & \text{on } D_\alpha^{k+1} \text{ for } \alpha \in C_2^{k+1} \\ \tilde{s}^k & \text{on } B^k \end{cases}$$

It is clear that $\hat{s}^{k+1}$ is continuous and we check as before that for every $x \in \partial D_\alpha^{k+1}$ we have $\hat{s}^{k+1}(x) = \hat{s}^{k+1}(\varphi_\alpha^{k+1}(x))$. Thus $\hat{s}^{k+1}$ induce a continuous map $\tilde{s}^{k+1} : B^{k+1} \longrightarrow E$ and it is straightforward to check that this is indeed a section that extends $s_{A^{k+1}} : A^{k+1} \longrightarrow E$. This achieves the proof. $\square$

We are now ready to prove the main theorem :

Theorem 4.0.1. *Suppose that* $EG \xrightarrow{\pi} BG$ *is a universal principal* G*-bundle and* X *is a CW-complex. Then the correspondance* $[X, BG] \longrightarrow \mathrm{Prin}_G(X)$ *given by* $[f] \mapsto [f^*(EG)]$ *is bijective.*

Proof. We start by showing that the map is surjective, so consider a principal G-bundle $P \xrightarrow{\pi} X$ over X, we must construct a map $f : X \longrightarrow BG$ such that $P \simeq f^*(EG)$. This is equivalent to the construction of G-equivariant map over the identity map, i.e :

$$\begin{array}{ccc} P & \xrightarrow{\simeq} & f^*(EG) \\ {\scriptstyle \pi_P}\downarrow & & \downarrow \\ X & \xrightarrow{\mathrm{Id}} & X \end{array} \tag{5}$$

which is in turn equivalent to finding a G-equivariant map $\phi : P \longrightarrow EG$ and putting $f : X \longrightarrow BG$ the induced base-map as shown in the following diagram :

$$\begin{array}{ccc} P & \xrightarrow{\phi} & EG \\ {\scriptstyle \pi_P}\downarrow & & \downarrow \\ X & \xrightarrow{f} & BG \end{array} \tag{6}$$

But this is the same (according to corollary (4.0.1)) as finding a section of the associated bundle $P \times_G EG \longrightarrow X$, the preceding lemma guarantees the existence of such a section since EG is weakly contractible. This proves the surjectivity.

To prove the injectivity, let $f_0, f_1 : X \longrightarrow BG$ be two maps with isomorphic pullbacks, i.e $f_0^*(EG) \simeq f_1^*(EG)$, we will show that f_0 and f_1 must be homotopic. The map $f_0 : X \longrightarrow BG$ induce a morphism of principal G-bundles via the pullback diagram and the isomorphism $f_0^*(EG) \simeq f_1^*(EG)$ is equivalent to morphism of principal G-bundles $\phi_1 : f_0^*(EG) \longrightarrow EG$ over $f_1 : X \longrightarrow BG$, this can be summarized in the following diagrams :

$$\begin{array}{ccc} f_0^*(EG) & \xrightarrow{\phi_0} & EG \\ {\scriptstyle \pi_0}\downarrow & & \downarrow{\scriptstyle \pi} \\ X & \xrightarrow{f_0} & BG \end{array} \qquad \begin{array}{ccc} f_0^*(EG) & \xrightarrow{\phi_1} & EG \\ {\scriptstyle \pi_0}\downarrow & & \downarrow{\scriptstyle \pi} \\ X & \xrightarrow{f_1} & BG \end{array} \tag{7}$$

Denote $s_0, s_1 : X \longrightarrow f_0^*(EG) \times_G EG$ the corresponding sections, which according to the proof of proposition (4.0.2) are defined by the relation :

$$s_0(\pi_0(Z)) = [z, \phi_0(z)] \quad \text{and} \quad s_1(\pi_0(z)) = [z, \phi_1(z)]. \tag{8}$$

Put $P := f_0^*(EG) \times I$ and $X_t = X \times \{t\}$. We can view s_i as a section $s_i \in \Gamma(X_i, P \times_G EG)$ for $i = 0, 1$. Thus we can define $s_0 \sqcup s_1 \in \Gamma(X_0 \sqcup X_1, P \times_G EG)$ by the formula :

$$(s_0 \sqcup s_1)_{|X_i} = s_i, \qquad i = 0, 1.$$

Now since $(X \times I, X_0 \sqcup X_1)$ is a CW-pair and EG is weakly contractible, we obtain according to the preceding lemma that $s_0 \sqcup s_1$ can be extended to a section $s \in \Gamma(X \times I, P \times_G EG)$. Denote $\phi : P \longrightarrow EG$ the morphism corresponding to s via the relation $s(\pi(z), t) = [(z, t), \phi(z, t)]$ and consider the commutative diagram :

$$\begin{array}{ccc} P & \xrightarrow{\phi_1} & EG \\ {\scriptstyle (\pi_0, \mathrm{Id})}\downarrow & & \downarrow{\scriptstyle \pi} \\ X \times I & \xrightarrow{F} & BG \end{array} \tag{9}$$

Since by definition $s(\pi_0(z), 0) = s_0(\pi_0(z))$ and $s(\pi_0(z), 1) = s_1(\pi_0(z))$, we obtain from relation (8) that :

$$\phi(z, 0) = \phi_0(z) \quad \text{and} \quad \phi(z, 1) = \phi_1(z).$$

Thus for all $x \in X$, we have that $F(x, 0) = f_0(x)$ and $F(x, 1) = f_1(x)$ which gives that f_0 and f_1 are homotopic maps. □

The space BG will be called a *classifying* space for the group G, and if $P \xrightarrow{\pi} X$ is a principal G-bundle, any map $f : X \longrightarrow BG$ such that $P \simeq f^*(EG)$ will be called a *classifying map* for P.

Now we address the uniqueness of the universal principal G-bundle. We start with the statement of a popular theorem called *the long exact sequence of a fibration* :

Theorem 4.0.2. *Let $E \xrightarrow{\pi} B$ be a fiber bundle with fiber type F. Then there exists a long exact sequence of homotopy groups :*

$$\cdots \longrightarrow \pi_n(F) \to \pi_n(E) \to \pi_n(B) \to \pi_{n-1}(F) \to \cdots \to \pi_0(F) \to \pi_0(E) \to \pi_0(B).$$

A first consequence of this theorem is that BG can be taken to have a relatively simple topological structure :

Corollary 4.0.2. *The classifying space BG can be taken to have a CW-complex structure.*

Proof. Let $EG \xrightarrow{\pi} BG$ be any universal principal G-bundle and consider a CW-approximation $\phi : BG' \longrightarrow BG$ of BG. Then we have a pullback diagram :

$$\begin{array}{ccc} \phi^*(EG) & \xrightarrow{\text{pr}_2} & EG \\ {\scriptstyle \text{pr}_1}\downarrow & & \downarrow{\scriptstyle \pi} \\ BG' & \xrightarrow{F} & BG \end{array} \tag{10}$$

To be able to show that BG can be replaced by the CW-complex BG', one must show that the principal G-bundles $\phi^*(EG) \xrightarrow{\text{pr}_1} BG'$ is universal, i.e $\phi^*(EG)$ is a contractible space. This can be done by considering the long exact homotopy sequences stated in the previous theorem which along with the pullback diagram gives the following commutative diagram :

$$\begin{array}{ccccccccccc} \ldots & \longrightarrow & \pi_n(G) & \longrightarrow & \pi_n(EG) & \longrightarrow & \pi_n(BG) & \longrightarrow & \pi_{n-1}(G) & \longrightarrow & \ldots \\ & & \downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle \text{pr}_2^*} & & \downarrow{\scriptstyle \phi^*} & & \downarrow{\scriptstyle \simeq} & & \\ \ldots & \longrightarrow & \pi_n(G) & \longrightarrow & \pi_n(\phi^*EG) & \longrightarrow & \pi_n(BG') & \longrightarrow & \pi_{n-1}(G) & \longrightarrow & \ldots \end{array} \tag{11}$$

Since $\phi^* : \pi_n(BG) \longrightarrow \pi_n(BG')$ is an isomorphism, we obtain by the five lemma that $\pi_n(EG) \simeq \pi_n(\phi^*EG)$ for all $n \in \mathbb{N}$, which shows that $\phi^*(EG)$ is weakly contractible and achieves the proof. ☐

We suppose in what follows that the classifying space BG always admits a CW-complex structure. We now state the "uniqueness" theorem for universal principal G-bundles.

Theorem 4.0.3. *Let $EG \longrightarrow BG$ and $E'G \longrightarrow B'G$ be two universal principal G-bundles. There exists a homotopy equivalence $B'G \longrightarrow BG$ that is covered by a G-equivariant homotopy equivalence $E'G \longrightarrow EG$. In this sense, universal principal G-bundles are unique up to homotopy equivalence.*

Proof. Choose two classifying maps $f : B'G \longrightarrow BG$ and $g : BG \longrightarrow B'G$ such that $E'G \simeq f^*(EG)$ and $EG \simeq g^*(E'G)$. Then the composite map $f \circ g : BG \longrightarrow BG$ is a classifying map of EG itself since :

$$(f \circ g)^*(EG) \simeq g^*(f^*(EG)) \simeq g^*(E'G) \simeq EG.$$

Thus by the injectivity of the map $[BG, BG] \longrightarrow \mathrm{Prin}_G(BG)$ we obtain that $f \circ g$ must be homotopic to Id_{BG}. Similarly, we show that $g \circ f$ is homotopic to $\mathrm{Id}_{B'G}$. Hence $f : B'G \longrightarrow BG$ is a homotopy equivalence. □

We thus have a well-defined correspondence $G \mapsto BG$ from the category of topological groups to the category of homotopy classes of CW-complexes, which is functorial according to the following theorem :

Theorem 4.0.4. *To each homomorphism of topological groups $\phi : G \longrightarrow H$ is associated a natural homotopy class $B\phi \in [BG, BH]$ such that if $\phi \in \mathrm{Hom}(G, H)$ and $\psi \in \mathrm{Hom}(H, K)$ then $[B(\phi \circ \psi)] = [B\phi \circ B\psi]$ and $B\mathrm{Id} = \mathrm{Id}$.*

Proof. We begin by the construction of $B\phi : BG \times BH$. The group homomorphism $\phi : G \longrightarrow H$ induce a topological action $\rho_\phi : G \longrightarrow \mathrm{Aut}(H)$ given by $g \mapsto r_{\phi(g)}$ so that the associated bundle $EG \times_{\rho_\phi} H$ is a principal H-bundle over BG and thus corresponds to a classifying map $B\phi : BG \longrightarrow BH$, i.e :

$$(B\phi)^*(EH) \simeq EG \times_{\rho_\phi} H.$$

Now we check the functoriality of B. If $\phi = \mathrm{Id}$ then $\rho_\phi : G \longrightarrow \mathrm{Aut}(G)$ is just the trivial action $g \mapsto r_g$ and thus $EG \times_{\rho_\phi} G \simeq EG \simeq (B\phi)^*(EG)$ hence we get $[B\phi] = [\mathrm{Id}_{BG}]$. Next, select two group homomorphisms $\phi : G \longrightarrow H$ and $\psi : H \longrightarrow K$, then the isomorphism :

$$EG \times_{\rho_{\psi \circ \phi}} K \simeq (EG \times_{\rho_\phi} H) \times_{\rho_\psi} K,$$

gives that :

$$B(\psi \circ \phi)^*(EK) \simeq (B\phi)^* \Big((B\psi)^*(EK) \Big) \simeq (B\psi \circ B\phi)^*(EK).$$

Thus $[B(\psi \circ \phi)] = [B\psi \circ B\phi]$ and we conclude that B is a functor. □

We achieve this section with a result on the universal bundle associated to a subgroup of a given topological group :

Proposition 4.0.3. *Let $H \overset{\iota}{\hookrightarrow} G$ be an inclusion of topological groups such that the canonical projection $G \longrightarrow G/H$ is a principal H-bundle. Then we can take $EH = EG$ and $BH = EG \times_G (G/H)$.*

In particular if G is a Lie group and H is a closed subgroup, then the canonical projection $G \longrightarrow G/H$ is always a principal H-bundle and thus the preceding result holds in this case.

5. Existence and explicit examples of universal principal bundles

We begin by presenting some specific situations where we can determine explicitly the universal principal bundle :

1. The easiest example is that of $G = \mathbb{Z}$ since the universal principal $\mathbb{Z}$-bundle is just the universal cover $\mathbb{R} \xrightarrow{\pi} S^1$ since $\mathbb{R}$ is a contractible space.

2. A non-trivial situation occurs in the case $G = \mathbb{Z}_2$: We begin by defining the infinite unit sphere $S^\infty = \underset{n \mapsto +\infty}{\operatorname{colim}} S^n$, we have that :

Lemma 5.0.1. *The infinite unit sphere S^∞ is contractible.*

Proof. Denote $\mathbb{R}^\infty = \operatorname{colim}_{n \to \infty} \mathbb{R}^n$ the vector space of infinite sequences $x = (x_n)_{n \in \mathbb{N}}$ whose terms vanish starting from a given rank. We can check that $\mathbb{R}^\infty$ is a Hilbert space for the norm :

$$\|x\| = \left(\sum_{n \in \mathbb{N}} x_n^2 \right)^{\frac{1}{2}}.$$

The infinite sphere S^∞ can be viewed as the unit sphere of $\mathbb{R}^\infty$, i.e :

$$S^\infty = \{x \in \mathbb{R}^\infty,\ \|x\| = 1\}.$$

Put $e = (1, 0, \dots) \in S^\infty$. To show that S^∞ is contractible we must construct a homotopy between Id_{S^∞} and the constant map e, to do this we start by defining the linear map $f : \mathbb{R}^\infty \longrightarrow \mathbb{R}^\infty$ given by the formula :

$$f(x_1, x_2, \dots) = (0, x_1, x_2, \dots).$$

Since $\|f(x)\|= \|x\|$, we get that f is continuous and that it induces a continuous map $f : S^\infty \longrightarrow S^\infty$. Now f is homotopic to the identity, to see this define the map $H : S^\infty \times I \longrightarrow S^\infty$ by the formula :

$$H(x,t) = \frac{tf(x) + (1-t)x}{\|tf(x) + (1-t)x\|}.$$

This is well defined since $\{x, f(x)\}$ is always linearly independent. Furthermore it is clear that H is continuous and that $H(x,0) = x$ and $H(x,1) = f(x)$. Hence H is a homotopy between f and the identity map. Now the concluding remark is that f is also homotopic to e via the map $\hat{H} : S^\infty \times I \longrightarrow S^\infty$ given by :

$$\hat{H}(x,t) = \frac{te + (1-t)f(x)}{\|te + (1-t)f(x)\|}.$$

We thus obtain that Id_{S^∞} is homotopic to e which achieves the proof. □

Now define $\mathbb{R}P^\infty = \underset{n\mapsto+\infty}{\mathrm{colim}} \mathbb{R}P^n$, and consider the action :

$$\mathbb{Z}_2 \times S^\infty \longrightarrow S^\infty, \quad ([n], x) \mapsto (-1)^n x.$$

This is clearly continuous since its restriction to $\mathbb{Z}_2 \times S^n$ is continuous and S^∞ is the colimit of the unit spheres S^n. Since $\mathbb{Z}_2$ is compact, the canonical projection $\pi : S^\infty \longrightarrow S^\infty/\mathbb{Z}_2$ defines a principal $\mathbb{Z}_2$-bundle. Let's inspect further $S^\infty/\mathbb{Z}_2$, from $\pi(S^n) = \mathbb{R}P^n$ we get that $S^\infty/\mathbb{Z}_2 = \cup_{n\in\mathbb{N}}\mathbb{R}P^n$. Moreover, a subset $U \subset S^\infty/\mathbb{Z}_2$ is open if and only if $\pi^{-1}(U) \cap S^n$ is open in S^n which is equivalent to $U \cap \mathbb{R}P^n$ being open in $\mathbb{R}P^n$, thus we obtain that :

$$S^\infty/\mathbb{Z}_2 = \underset{n\mapsto+\infty}{\mathrm{colim}} \mathbb{R}P^n = \mathbb{R}P^\infty.$$

In summary we conclude that $S^\infty \xrightarrow{\pi} \mathbb{R}P^\infty$ is a universal principal $\mathbb{Z}_2$-bundle.

3. The infinite dimensional sphere S^∞ can also be viewed as the colimit of odd dimensional spheres $S^{2n+1} \subset \mathbb{C}^{n+1}$. Define the *infinite complex projective space* to be the colimit space :

$$\mathbb{C}P^\infty = \underset{n\mapsto+\infty}{\mathrm{colim}} \mathbb{C}P^n.$$

We can show as before that the principal S^1-bundles $S^{2n+1} \longrightarrow \mathbb{C}P^\infty$ define by taking the colimit a principal S^1-bundle $S^\infty \longrightarrow \mathbb{C}P^\infty$ which is universal since S^∞ is contractible.

4. We proceed to the calculation of the universal principal $\mathrm{U}(n)$-bundle (resp. $\mathrm{O}(n)$-bundle) : Start by defining the infinite dimensionalStiefel manifold $\mathrm{St}_{\mathbb{K}}(k,\infty) = \underset{n\mapsto+\infty}{\mathrm{colim}}\mathrm{St}_{\mathbb{K}}(k,n)$ ($\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$). Then we have that :

Lemma 5.0.2. *The infinite dimensional Stiefel manifold* $\mathrm{St}_{\mathbb{K}}(k,\infty)$ *is contractible.*

Proof. We show this for $\mathbb{K}=\mathbb{R}$, the case $\mathbb{K}=\mathbb{C}$ is similar. Endow $\mathbb{R}^\infty$ with the scalar product $\langle\ ,\ \rangle$ given by :

$$\langle v,w\rangle = \sum_{n\in\mathbb{N}} v_n w_n.$$

Then $\mathrm{St}_{\mathbb{R}}(k,\infty)$ might be viewed as the space of k-tuples $(v_1,\dots,v_k)$ such that $v_i\in\mathbb{R}^\infty$ and $\langle v_i,v_j\rangle=\delta_{ij}$. Define the shift operator $f:\mathbb{R}^\infty\longrightarrow\mathbb{R}^\infty$ as the linear map given by

$$f(x_1,x_2,\dots)=(0,x_1,x_2,\dots).$$

This induce a map $f:\mathrm{St}_{\mathbb{R}}(k,\infty)\longrightarrow\mathrm{St}_{\mathbb{R}}(k,\infty)$. We claim that the identity $\mathrm{Id}_{\mathrm{St}_{\mathbb{R}}(k,\infty)}$ is homotopic to f^k. Indeed, define the continuous map $H:\mathrm{St}_{\mathbb{R}}(k,\infty)\times I\longrightarrow\mathrm{St}_{\mathbb{R}}(k,\infty)$ given for $v=(v_1,\dots,v_k)$ by the formula :

$$H(v,t)=\mathrm{Gram}(tf^k(v_1)+(1-t)v_1,\dots,tf^k(v_k)+(1-t)v_k),$$

where Gram denotes the Gram-Schmidt orthogonalization procedure. It is easily checked that $H(v,0)=v$ and $H(v,1)=f^k(v)$, which gives the desired homotopy. Now, f^k is also homotopic to the point $(e_1,\dots,e_k)$ via the homotopy $\hat{H}:\mathrm{St}_{\mathbb{R}}(k,\infty)\times I\longrightarrow\mathrm{St}_{\mathbb{R}}(k,\infty)$ given by the expression :

$$\hat{H}(v,t)=\mathrm{Gram}(te_1+(1-t)f^k(v_1),\dots,te_k+(1-t)f^k(v_k)).$$

We conclude that $\mathrm{Id}_{\mathrm{St}_{\mathbb{R}}(k,\infty)}$ is homotopic to $(e_1,\dots,e_k)$, and thus $\mathrm{St}_{\mathbb{R}}(k,\infty)$ is contractible. Essentially the same reasoning shows that $\mathrm{St}_{\mathbb{C}}(k,\infty)$ is contractible. □

We define the infinite dimensional Grassamannian by :

$$\mathrm{Gr}_{\mathbb{K}}(k,\infty)=\underset{n\mapsto+\infty}{\mathrm{colim}}\mathrm{Gr}_{\mathbb{K}}(k,n).$$

By taking the colimit of the principal $\mathrm{U}(n)$-bundles $\mathrm{St}_{\mathbb{C}}(k,n) \xrightarrow{\pi} \mathrm{Gr}_{\mathbb{C}}(k,n)$ we get a principal $\mathrm{U}(n)$-bundle $\mathrm{St}_{\mathbb{C}}(k,\infty) \xrightarrow{\pi} \mathrm{Gr}_{\mathbb{C}}(k,\infty)$ which is universal according to the preceding lemma.
Similarly, for $\mathbb{K}=\mathbb{R}$ we can show that $\mathrm{St}_{\mathbb{R}}(k,\infty)$ is contractible, and thus we obtain that the universal principal $\mathrm{O}(n)$-bundle is just the (colimit) principal $\mathrm{O}(n)$-bundle $\mathrm{St}_{\mathbb{R}}(k,\infty) \xrightarrow{\pi} \mathrm{Gr}_{\mathbb{R}}(k,\infty)$.

5. Since every compact Lie group G admits a faithful representation in some unitary group $\mathrm{U}(k)$, the preceding example and proposition (4.0.3) shows that we can take $EG = \mathrm{St}_{\mathbb{C}}(k,\infty)$ and $BG = E\mathrm{U}(k) \times_{\mathrm{U}(k)} (\mathrm{U}(k)/G)$. In summary, we have shown the existence of the universal principal G-bundle for every compact Lie group G.

6. Universal vector bundles

We know from theorem (2.0.2) that when $E \xrightarrow{\pi} B$ is a vector bundle and $f_0, f_1 : X \longrightarrow B$ are homotopic maps, then we get that $f_0^*(E) \simeq f_1^*(E)$ as vector bundles. Thus if we denote $\mathrm{Vect}_k(X,\mathbb{K})$ the space of isomorphism classes $\mathbb{K}$-vector bundles over X of rank k, we obtain a well-defined correspondance :

$$[X,B] \longrightarrow \mathrm{Vect}_k(X,\mathbb{K}), \quad [f] \mapsto [f^*E].$$

So the question that arise naturally at this point is *whether there exists a "universal" vector bundle $E \xrightarrow{\pi} B$ such that the map the preceding correspondance is bijective.* We show in what follows that this question admits an affirmative answer when X is a CW-complex. The proof of this claim will use the results of the preceding section which sort of ease the task. Note that the result still holds in the more general case where X is a paracompact Hausdorff space, for a complete proof that is independent of principal bundles one might consult [Hat03].

We start we the case $\mathbb{K}=\mathbb{R}$, the proof for $\mathbb{K}=\mathbb{C}$ is analogous. Suppose that X is a CW-complex and let $G=\mathrm{O}(k)$, we have seen that we can take $BG = \mathrm{Gr}_{\mathbb{R}}(k,\infty)$ and $EG = \mathrm{St}_{\mathbb{R}}(k,\infty)$. Recall that the map :

$$\alpha : [X, \mathrm{Gr}_{\mathbb{R}}(k,\infty)] \longrightarrow \mathrm{Prin}_{\mathrm{O}(k)}(X), \quad [f] \mapsto [f^*EG],$$

is a bijective correspondence. Now define the associated bundle

$$E_k(\mathbb{R}) := \mathrm{EG} \times_{\mathrm{O}(k)} \mathbb{R}^k.$$

This is a rank k (real) vector bundle over $\mathrm{Gr}_{\mathbb{R}}(k,\infty)$. The main theorem of this part can be stated as follows :

Theorem 6.0.1. *Suppose that X is a CW-complex , then the map :*

$$\beta : [X, \mathrm{Gr}_{\mathbb{R}}(k,\infty)] \longrightarrow \mathrm{Vect}_k(X,\mathbb{R}), \quad [f] \mapsto [f^*(E_k(\mathbb{R}))],$$

is a bijective correspondence.

Proof. The key point is to recall that over a paracompact space, the operation of association :

$$\mathrm{Prin}_{\mathrm{O}(k)}(X) \longrightarrow \mathrm{Vect}_k(X,\mathbb{R}), \quad P \mapsto P \times_{\mathrm{O}(k)} \mathbb{R}^k,$$

is bijective. Next, it is straightforward to check that the following diagram is commutative :

$$\begin{array}{ccc} [X, \mathrm{Gr}_{\mathbb{R}}(k,\infty)] & \xrightarrow{\ \alpha\ } & \mathrm{Prin}_{\mathrm{O}(k)}(X) \\ & \searrow^{\beta} & \downarrow \simeq \\ & & \mathrm{Vect}_k(X,\mathbb{R}) \end{array} \tag{12}$$

i.e $f^*(EG \times_{\mathrm{O}(k)} \mathbb{R}^k) \simeq f^*(EG) \times_{\mathrm{O}(k)} \mathbb{R}^k$. Thus we conclude that β is bijective. □

Now for $\mathbb{K} = \mathbb{C}$ we let $G = \mathrm{U}(k)$, we have seen in the preceding section that it is possible to set $BG = \mathrm{Gr}_{\mathbb{C}}(k,\infty)$ and $EG = \mathrm{St}_{\mathbb{C}}(k,\infty)$. We put $E_k(\mathbb{C}) = EG \times_{\mathrm{U}(k)} \mathbb{C}^k$, and by analogy to the former case we get the following statement :

Theorem 6.0.2. *Suppose that X is a CW-complex , then the map :*

$$\beta : [X, \mathrm{Gr}_{\mathbb{C}}(k,\infty)] \longrightarrow \mathrm{Vect}_k(X,\mathbb{C}), \quad [f] \mapsto [f^*(E_k(\mathbb{C}))],$$

is a bijective correspondence.

We call $E_k(\mathbb{K}) \longrightarrow \mathrm{Gr}_{\mathbb{K}}(k,\infty)$ *a universal $\mathbb{K}$-vector bundle of rank k*. Given a rank k vector bundle $E \xrightarrow{\pi_E} X$, a map $f : X \longrightarrow \mathrm{Gr}_{\mathbb{K}}(k,\infty)$ such that $E \simeq f^*(E_k(\mathbb{K}))$ will be called *a classifying map for E*.

In the final part of this section, we specialize in the case where X is a differentiable manifold which we will denote by the letter M. We show that for an important class of differentiable manifolds (finite type manifolds), *there is a classification theorem for vector bundles such that the corresponding universal bundle is a finite dimensional differentiable vector bundle.* Before delving into this, let us recall some elementary facts about manifolds :

Definition 6.0.1. *A good cover of a differentiable manifold M is any open cover whose elements are contractible and such that finite intersections of its elements are also contractible.*

A well-known result about good covers is the following :

Theorem 6.0.3. *Any differentiable manifold admits a good cover.*

We say that a differentiable manifold M is of *finite type* if it admits a *finite* good cover. In particular, compact differentiable manifolds are of finite type. We will show that for differentiable manifolds of finite type, the universal vector bundle is a (finite dimensional) differentiable vector bundle. As usual, we treat the case $\mathbb{K} = \mathbb{R}$ since the complex case is identical.

Denote $V \xrightarrow{\pi} \mathrm{Gr}_{\mathbb{R}}(k,n)$ the trivial bundle of rank n over the Grassmannian $\mathrm{Gr}_{\mathbb{R}}(k,n)$, i.e $V = \mathrm{Gr}_{\mathbb{R}}(k,n) \times \mathbb{R}^n$ and define :

$$E_{k,n}(\mathbb{R}) = \{(v,\ell) \in V,\ v \in \ell\}.$$

It is straightforward to check that $E_{k,n}(\mathbb{R}) \xrightarrow{\pi} \mathrm{Gr}_{\mathbb{R}}(k,n)$ is vector subbundle of V of rank k over the Grassmannian $\mathrm{Gr}_{\mathbb{R}}(k,n)$ called *the tautological bundle of rank k.* We start with a lemma :

Lemma 6.0.1. *Let $E \xrightarrow{\pi_E} M$ be a vector bundle over a differentiable manifold of finite type. There exists on M finitely many continuous sections of E which span the fiber at each point.*

Proof. Let $U_1, \ldots, U_r$ be a finite good cover of M. Since U_i is contractible, we get that $E_{|U_i}$ is trivial and so we can find k sections $\sigma_{i,1}, \ldots, \sigma_{i,k}$ over each U_i which form a basis of the fiber above any point in U_i. Next choose open sets $V_1, \ldots, V_r$ of M such that $\bar{V}_i \subset U_i$ and continuous functions $f_i : M \longrightarrow \mathbb{R}$ such that $f_i = 1$ on V_i and $f_i = 0$ outside of U_i (this is possible since M is paracompact), we thus obtain that :

$$\{f_i\sigma_{i,1}, \ldots, f_i\sigma_{i,k},\ 1 \leq i \leq r\},$$

is a family of global sections of E which span the fiber at every point. □

Proposition 6.0.1. *Let E be a rank k (real) vector bundle over a differentiable manifold M. Suppose there are n global sections $\sigma_1, \ldots, \sigma_n$ of E which span the fiber at every point. Then there is a map $f_E : M \longrightarrow \mathrm{Gr}_{\mathbb{R}}(k, n)$ such that $E \simeq f_E^*(E_{k,n}(\mathbb{R}))$.*

Proof. Endow $\mathbb{R}^n$ with its usual scalar product $\langle\ ,\ \rangle$ and define for each $p \in M$ the evaluation map, which is the linear map given by :

$$\mathrm{ev}_p : \mathbb{R}^n \longrightarrow E_p, \quad \mathrm{ev}_p(e_i) = \sigma_i(p).$$

Now write $\mathbb{R}^n = \ker(\mathrm{ev}_p) \overset{\perp}{\oplus} S_p$. Since $\mathrm{rg}(\mathrm{ev}_p) = k$ we obtain that $\dim(S_p) = k$ and thus $S_p \in \mathrm{Gr}_{\mathbb{R}}(k, n)$. Next we consider the map :

$$f : M \longrightarrow \mathrm{Gr}_{\mathbb{R}}(k, n), \quad p \mapsto S_p.$$

We start by checking that f is a continuous map :

Let $p \in M$, we can suppose without loss of generality that $\{\sigma_1(p), \ldots, \sigma_k(p)\}$ is a basis for E_p. Since the sections σ_i are continuous, there exists an open neighborhood U of p in M such that $\{\sigma_1(q), \ldots, \sigma_k(q)\}$ is a basis for E_q for all $q \in U$. Now define the local trivialization :

$$\phi : U \times \mathbb{R}^k \longrightarrow E_{|U}, \quad (p, e_i) \mapsto \sigma_i(q),$$

and write for all $k + 1 \leq i \leq n$:

$$\phi_p^{-1}(\sigma_i(q)) = \sum_{j=1}^{k} a_{ij}(q) e_j,$$

where $a_{ij} : U \longrightarrow \mathbb{R}$ are continuous functions. If we view $\mathbb{R}^k$ as the subspace of $\mathbb{R}^n$ spanned by the first k vectors of the canonical basis, is becomes clear that $\ker(\mathrm{ev}_q) = \ker \phi_q^{-1} \circ \mathrm{ev}_q$ and an easy calculation leads to :

$$(\phi_q^{-1} \circ \mathrm{ev}_q)(e_i) = \begin{cases} e_i & \text{if } 1 \leq i \leq k \\ \sum_{j=1}^{k} a_{ij}(q) e_j & \text{if } k + 1 \leq i \leq n \end{cases}$$

Thus $\mathcal{B}_q = \{e_i - \sum_{j=1}^{k} a_{ij}(q) e_j,\ k + 1 \leq i \leq n\}$ is contained in $\ker(\mathrm{ev}_q)$, and since it is clearly a free family of $(n - k)$ elements, it is a basis of $\ker(e_v)_q$. Now using the Gram-Schmidt process, we complete $\mathcal{B}_q$ into an orthonormal basis $\{f_1(q), \ldots, f_k(q)\}$ of S_q, and the so obtained function $f_i : U \longrightarrow \mathbb{R}^n$ are continuous. This gives a continuous function :

$$g : U \longrightarrow \mathrm{St}_{\mathbb{R}}(k, n), \quad q \mapsto (f_1(q), \ldots, f_k(q)),$$

which in turn gives a commutative diagram :

$$\begin{array}{ccc} & & \mathrm{St}_{\mathbb{R}}(k,n) \\ & \overset{g}{\nearrow} & \downarrow \pi \\ U & \xrightarrow[f_{|U}]{} & \mathrm{Gr}_{\mathbb{R}}(k,n) \end{array} \tag{13}$$

This shows that $f_{|U} : U \longrightarrow \mathrm{Gr}_{\mathbb{R}}(k, n)$ is continuous, and since U was an arbitrary neighborhood we get that f is globally continuous. Finally, we check that $E \simeq f^*(E_{k,n}(\mathbb{R}))$:

Notice that since $E_{k,n}(\mathbb{R}) \xrightarrow{\pi} \mathrm{Gr}_{\mathbb{R}}(k, n)$ has fiber over S_p equal to S_p itself, we obtain that $f^*(E_{k,n}(\mathbb{R})) \xrightarrow{\pi} M$ is a vector bundle over M with fiber over p equal to S_p. Next define the map $I : f^*(E_{k,n}(\mathbb{R})) \longrightarrow E$ such that $I_{|S_p} := \mathrm{ev}_{p|S_p} : S_p \longrightarrow E_p$. Then I satisfies the commutative diagram :

$$\begin{array}{ccc} E & \xrightarrow{I} & f^*(E_{k,n}(\mathbb{R})) \\ \pi_E \downarrow & & \downarrow \pi \\ M & \xrightarrow[\mathrm{Id}_M]{} & M \end{array} \tag{14}$$

Furthermore, I induce an isomorphism on the fiber by its definition. To verify that it is an isomorphism of vector bundles, it is thus sufficient to check that it is continuous. □

The map $f_E : M \longrightarrow \mathrm{Gr}_{\mathbb{R}}(k, n)$ such that $f^*(E_{k,n}(\mathbb{R})) \simeq E$ is called a *classifying map for the vector bundle* E over M. We now give a restatement of the preceding result in the style of a "classification theorem", to do this we will call a manifold *of type* r if it admits a good cover with r contractible open sets. The proof of lemma (6.0.1) shows that any rank k vector bundle

E over a type r manifold admits $n = kr$ global sections which span the fiber at every point. Thus we get the following theorem :

Theorem 6.0.4. *Let M be a differentiable manifold and type r. Then for every $n \geq kr$, the map :*

$$\beta_n : [M, \mathrm{Gr}_{\mathbb{R}}(k,n)] \longrightarrow \mathrm{Vect}_k(M, \mathbb{R}) \quad [f] \mapsto f^*(E_{k,n}(\mathbb{R})).$$

is a surjective correspondence.

One might of course expect based on the previous study that the map β_n is injective : isomorphic pullbacks $f^*(E_{k,n}(\mathbb{R})) \simeq g^*(E_{k,n}(\mathbb{R}))$ give rise to homotopic maps $f \simeq g$, but this fails to be true. However the statement might still have some truth in it and a weaker version of it is given by the following theorem which can be consulted in [Spi79] and [Osb83] :

Theorem 6.0.5. *Let $f, g : M \longrightarrow \mathrm{Gr}_{\mathbb{R}}(k,n)$ be two maps such that :*

$$f^*(E_{k,n}(\mathbb{R})) \simeq g^*(E_{k,n}(\mathbb{R})).$$

Consider the natural inclusion $\alpha : \mathrm{Gr}_{\mathbb{R}}(k,n) \longrightarrow \mathrm{Gr}_{\mathbb{R}}(k,m)$ for $m \geq 2n$. Then $\bar{f} = \alpha \circ f$ and $\bar{g} = \alpha \circ g$ are homotopic in $\mathrm{Gr}_{\mathbb{R}}(k,m)$.

Proposition 6.0.2. *Let $E \xrightarrow{\pi_E} M$ be a vector bundle over a manifold a finite type, then any two classifying maps for E are homotopic.*

A concluding remark of this section is that the universal vector bundle $E_k(\mathbb{K}) \longrightarrow \mathrm{Gr}_{\mathbb{K}}(k,\infty)$ of rank k might be taken as the co-limit of the tautological bundles $E_{k,n}(\mathbb{K}) \longrightarrow \mathrm{Gr}_{\mathbb{K}}(k,n)$ as $n \mapsto \infty$, again one might want to see [Hat03].

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