

Pointwise bounds on the gradient and the spectrum of the Neumann-Poincaré operator: The case of 2 discs

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ABSTRACT. We compute the spectrum of the Neumann-Poincaré operator for two discs in \mathbb{R}^2 . We show how the behavior of the eigenvalues relates to $W^{1,\infty}$ estimates on the potential in 2D composites containing circular inclusions.

1. Introduction

This work is a contribution to the study of pointwise bounds on the gradients of solutions to elliptic PDE's in composite media made of inclusions embedded in a matrix phase. In mechanics, regions where inclusions touch or are close to touching are likely to concentrate stress, and therefore are likely to become preferred sites for the onset of fracture. Similarly, in optics, electromagnetic fields are likely to concentrate in narrow channels where the parameter contrast with surrounding regions is large, a fact that could be useful in applications such as microscopy, spectroscopy or bio-sensing. How do the sizes of the gradients depend on the geometry and of the coefficient contrasts is therefore an important question.

Over the last decade, this topic has inspired a number of mathematical works. In [8], the case of 2 circular inclusions separated by a distance δ was studied in the context of a conduction equation. Using the maximum principle, a $W^{1,\infty}$ bound independent of δ , was established on the potential. This result was extended to a general class of configurations by YanYan Li and M. Vogelius[15], who considered piecewise Hölder media: Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $C^{1,\alpha}$ boundary, which contains a finite number M of inclusions D_j with $C^{1,\alpha}$ boundary. Assume that the conductivity γ is $C^{0,\mu}$ in each inclusion and in $D_{M+1} = \Omega \setminus \cup_{j=1}^M \overline{D}_j$, and that $0 < \Lambda \leq \gamma(x) \leq \Lambda^{-1}$ in Ω . If $u \in H^1(\Omega)$ is a solution to

$$(1.1) \quad \operatorname{div}(\gamma(x)\nabla u(x)) = 0 \quad \text{in } \Omega,$$

then the following interior estimate holds for any $\varepsilon > 0$

$$\sum_{j=1}^{M+1} \|u\|_{C^{1,\alpha}(\overline{D}_j \cap \Omega_\varepsilon)} \leq C \|u\|_{L^2(\Omega)},$$

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where $\Omega_\varepsilon = \{x \in \Omega, \text{dist}(x, \partial\Omega) < \varepsilon\}$. The constant C in this estimate depends on $\Omega, \lambda, M, \mu, \alpha$ but is independent on the inter-inclusion distance. This result was later generalized to strongly elliptic systems and particularly to the system of elasticity by YanYan Li and L. Nirenberg [14].

The situation is different if the material coefficients are degenerate (perfectly conducting or insulating inclusions) where the gradients may blow up as the inclusions come to touching (see e.g. [8]). How the bounds depend on the inter-inclusion distance was explicated in [7], who studied the case of two perfectly conducting $C^{2,\alpha}$ inhomogeneities embedded in a domain $\Omega \subset \mathbb{R}^n$ of conductivity $\gamma = 1$. The gradient of the potential was shown to satisfy

$$(1.2) \quad \begin{cases} \|\nabla u\|_{L^\infty} \leq \frac{C}{\sqrt{\delta}} \|u\|_{L^2(\partial\Omega)} & \text{for } n = 2, \\ \|\nabla u\|_{L^\infty} \leq \frac{C}{\delta |\ln \delta|} \|u\|_{L^2(\partial\Omega)} & \text{for } n = 3, \\ \|\nabla u\|_{L^\infty} \leq \frac{C}{\delta} \|u\|_{L^2(\partial\Omega)} & \text{for } n = 4. \end{cases}$$

The case $n = 2$ was derived independently by Yun, using conformal mapping techniques [19].

Several works focus on particular geometrical configurations [5, 3, 2, 6, 9, 16]. There, the potential u may have a series representation that lends itself to asymptotic analysis, so that one can address the question of how the bounds blow up when both the inclusions come to touching and their conductivities degenerate. Optimal upper and lower bounds on the potential gradients were obtained in [5, 3] for nearly touching pairs of circular inclusions. Spherical inclusions were studied in [2].

In this work, we consider the situation of 2 circular inclusions D_1, D_2 at a distance δ from each other, embedded in \mathbb{R}^2 . To fix ideas, we assume that D_1, D_2 have the same radius $r = 1$ and are centered at the points $(1 + \delta, 0), (-1 - \delta, 0)$, respectively. We assume that the conductivity γ_δ is piecewise constant, with values $\gamma_\delta(x) = k, 0 < k < \infty, k \neq 1$, in the inclusions and $\gamma_\delta(x) = 1$ in $\mathbb{R}^2 \setminus \overline{D_1 \cup D_2}$. Given a function H harmonic in \mathbb{R}^2 , the potential u solves

$$(1.3) \quad \begin{cases} \text{div}(\gamma_\delta \nabla u) = 0 & \text{in } \mathbb{R}^2 \\ u - H \rightarrow 0 & \text{as } |x| \rightarrow \infty. \end{cases}$$

In [4], the function u is shown to decompose as a sum $u = u_r + u_s$ of a regular part u_r , the gradient of which remains bounded as $\delta \rightarrow 0$, and a singular part u_s . Reformulated in terms of the configuration described above, their result states that for some constants C_1, C_2, C_3 , independent of δ and k

$$(1.4) \quad \begin{cases} |\nabla u_s|^+(\pm\delta, 0) \geq C_1 \frac{|\nabla H(0) \cdot e_2|}{2k + \sqrt{\delta}} \\ \|\nabla u_s\|_{\infty, \Omega} \leq C_2 \frac{|\nabla H(0) \cdot e_2|}{2k + \sqrt{\delta}} \\ \|\nabla u_r\|_{\infty, \Omega} \leq C_3, \end{cases}$$

if $0 < k < 1$, while for $k > 1$ the estimates read

$$(1.5) \quad \begin{cases} |\nabla u_s|^+(\pm\delta, 0) & \geq C_1 \frac{|\nabla H(0) \cdot e_1|}{2k^{-1} + \sqrt{\delta}} \\ \|\nabla u_s\|_{\infty, \Omega} & \leq C_2 \frac{|\nabla H(0) \cdot e_1|}{2k^{-1} + \sqrt{\delta}} \\ \|\nabla u_r\|_{\infty, \Omega} & \leq C_3, \end{cases}$$

Further development led to obtaining an asymptotic expansion of the potential with an explicit characterization of the singular part [11].

In this work, we address the problem of bounding the gradient of u from the point of view of integral representations and spectral decompositions. In section 2, we consider an integral representation of u in terms of layer potentials $\varphi = (\varphi_1, \varphi_2)$ defined on the boundaries of the inclusions. They satisfy a system of integral equation of the form

$$(1.6) \quad (\lambda I - K^*)\varphi = \begin{pmatrix} \partial_{\nu_1} H \\ \partial_{\nu_2} H \end{pmatrix},$$

where $\lambda = \frac{k+1}{2(k-1)}$ and where K^* is a compact operator. We note that the contrast only enters the first part of the operator on the right-hand side, whereas the inter-inclusion distance only appears in K^* , which motivates our interest in the spectral decomposition of the latter operator. Indeed, the Neumann-Poincaré operator K^* has a spectral decomposition, albeit being non-self-adjoint. A new scalar product can be defined on $L^2(\partial D_1) \times L^2(\partial D_2)$ for which K^* becomes self-adjoint. This process of symmetrization is due to T. Carleman [10], and was further developed by M. G. Krein [13]. It is studied in [12] in the particular context of the Laplace operator. We note that an integral equation similar to (1.6) arises in the context of cloaking by a plasmonic annulus. It was studied in [1] using also a spectral decomposition of the Neumann-Poincaré operator. In section 3, we compute the eigenvalues of K_δ^* : we show that they split in 2 families λ_n^\pm , with corresponding eigenfunctions $\varphi_{n,A/B,\pm}$, which converge to $\pm 1/2$. Next, we solve the integral equation (1.6) using the spectral decomposition of K^* . We show that its solution φ has a series representation on the basis of eigenfunctions that converges pointwise. In section 4, we show that one can read off the blow-up rate of u from this expansion, and recover the estimates (1.4,1.5). Throughout the text, we denote $D_e = \mathbb{R}^2 \setminus \overline{\partial D_1 \cup \partial D_2}$, and $u^+(x) = \lim_{t \rightarrow 0^+} u(x + t\nu_i(x))$ and $u^-(x) = \lim_{t \rightarrow 0^-} u(x + t\nu_i(x))$, for $x \in \partial D_i$.

2. The system of integral equations

For $\delta > 0$, we represent u solution to (1.3) as

$$(2.1) \quad u(x) = H(x) + S\varphi(x) := H(x) + \begin{pmatrix} S_1 & 0 \\ 0 & S_2 \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix},$$

where S_i denotes the single layer potential operator on ∂D_i ,

$$S_i\varphi(x) = \frac{1}{2\pi} \int_{\partial D_i} \log(|x-y|) \varphi(y) d\sigma(y).$$

Expressing the transmission conditions satisfied by the solutions to (1.3) shows that the layer potential φ satisfies (1.6) with

$$K^* \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \begin{pmatrix} K_1^* & \frac{\partial}{\partial \nu_1} S_2 \\ \frac{\partial}{\partial \nu_2} S_1 & K_2^* \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix},$$

where the integral operators K_i^* are defined on $L^2(\partial D_i)$ by

$$K_i^* g(x) = \int_{\partial D_i} \frac{(x-y) \cdot \nu_i(x)}{|x-y|^2} g(y) d\sigma(y).$$

Classical integral operator theory shows that when $\delta > 0$, the solution of this system is uniquely defined and is smooth. Indeed, each operator $K_i^* : H^s(\partial D_i) \rightarrow H^s(\partial D_i)$ is compact [18]. Further, since the kernels of the extradiagonal terms of K^* have the form

$$\frac{(x-y) \cdot \nu_i(x)}{|x-y|^2}, \quad (x,y) \in \partial D_1 \times \partial D_2 \quad \text{or} \quad (x,y) \in \partial D_2 \times \partial D_1,$$

their denominator is bounded below by δ , and so these terms are also compact. Classical potential theory applies to show that if $|\lambda| > 1/2$, then (1.6) has a unique solution $\varphi \in L^2(\partial D_1) \times L^2(\partial D_2)$, such that

$$\int_{\partial D_i} \varphi_i = 0, \quad i = 1, 2.$$

The operator K^* is not selfadjoint. Indeed, it is well known that the L^2 -adjoint of K_i^* is the operator K_i defined by

$$K_i g(x) = \int_{\partial D_i} \frac{(y-x) \cdot \nu_i(y)}{|x-y|^2} g(y) d\sigma(y),$$

and one easily checks that the adjoint of the extra-diagonal term $(\partial_{\nu_1} S_2)|_{\partial D_1}$ is

$$L_2 g(x) = \int_{\partial D_1} \frac{(y-x) \cdot \nu_1(y)}{|x-y|^2} g(y) d\sigma(y), \quad x \in \partial D_2,$$

and a similar expression for the adjoint L_1 of $(\partial_{\nu_2} S_1)|_{\partial D_2}$. The adjoint of K^* is thus

$$K \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \begin{pmatrix} K_1 & L_1 \\ L_2 & K_2 \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}.$$

From the Plemelj symmetrization principle, the operators K, K^* satisfy

$$SK^* = KS.$$

Since S is non-positive and self-adjoint (see lemma 2.1 in [12]), one can define a new inner-product on the space $L^2(\partial D_1) \times L^2(\partial D_2)$ by setting

$$(2.2) \quad \begin{aligned} \langle \varphi, \psi \rangle_S &:= \langle -S\varphi, \psi \rangle_{L^2} \\ &:= - \int_{\partial D_1} S\varphi_1 \psi_1 - \int_{\partial D_2} S\varphi_2 \psi_2, \end{aligned}$$

which turns K^* into a self-adjoint operator. However, since S is a pseudo-differential operator of order 1, the space $L^2(\partial D_1) \times L^2(\partial D_2)$ is not complete for this inner-product.

It follows from [12] that the spectrum of K^* is contained in $[-1/2, 1/2]$ and consists in a sequence of eigenvalues which converges to 0. Furthermore, the eigenvectors of K^* , including the null vectors, span $L^2(\partial D_1) \times L^2(\partial D_2)$. We introduce the weighted Sobolev space

$$\mathcal{W}^{1,-1}(\mathbb{R}^2) := \left\{ u : \frac{u(x)}{(1+|x|^2)^{1/2} \log(2+|x^2|)} \in L^2(\mathbb{R}^2), \nabla u \in L^2(\mathbb{R}^2) \right\}.$$

This space, which contains the constant functions, can be used to invert the Laplacian in the plane [17].

We first note that $\lambda = 1/2$ is an eigenvalue of K^* . Indeed, consider a solution to

$$\begin{cases} \nabla w = 0 & \text{in } \mathbb{R}^2 \setminus \overline{D_1 \cup D_2} \\ w = c_i & \text{on } D_i \quad i = 1, 2, \\ w = O(|x|^{-1}) & \text{as } |x| \rightarrow \infty, \end{cases}$$

where the constants c_1, c_2 are chosen so that

$$\int_{\partial D_1} \partial_\nu w + \int_{\partial D_2} \partial_\nu w = 0.$$

Using the conformal mapping described in section 3 below, w is in fact a multiple of the function $\log \left| \frac{x-a}{x+a} \right|$, with $a = \sqrt{\delta(2+\delta)}$. Since w is constant inside D_1 and D_2 , one easily obtain from the Plemelj formulas that

$$(2.3) \quad \varphi_0 = (\partial_\nu w|_{\partial D_1}^+, \partial_\nu w|_{\partial D_2}^+)$$

satisfies $K^* \varphi_0 = \frac{1}{2} \varphi_0$. If $\psi \in H^{-1/2}(\partial D_1) \times H^{-1/2}(\partial D_2)$ such that

$$\int_{\partial D_1} \psi_1 + \int_{\partial D_2} \psi_2 = 0,$$

were another eigenvector of K^* associated to the eigenvalue $1/2$, the function $v = S\psi$ would be harmonic in $\mathbb{R}^2 \setminus \overline{D_1 \cup D_2}$ and due to the Plemelj formulas, would be equal to constants C_1, C_2 on the discs D_1, D_2 . But then uniqueness of the solution to the Dirichlet problem in $W^{1,-1}(\mathbb{R}^2)$ (see [17]) would imply that $v = C_1 w + C_2 - \frac{C_1 c_2}{c_1}$ and in particular that $\partial_\nu v = C_1 \partial_\nu w$ on $\partial D_i, i = 1, 2$. It follows that $1/2$ is an eigenvalue of K^* and that the associated eigenspace has dimension 1.

The following proposition is easily deduced from the jump relations satisfied by the single layer potential:

PROPOSITION 2.1. Let $\lambda \in (-1/2, 1/2)$ be an eigenvalue of K^* and let $\psi = (\psi_1, \psi_2) \in H^{-1/2}(\partial D_1) \times H^{-1/2}(\partial D_2)$, such that

$$\int_{\partial D_1} \partial_\nu w + \int_{\partial D_2} \partial_\nu w = 0,$$

denote an associated eigenvector. Set

$$u(x) = S_1 \psi_1 + S_2 \psi_2.$$

Then $u \in W^{-1,1}(\mathbb{R}^2)$ and satisfies

$$(2.4) \quad \begin{cases} \Delta u &= 0 & \text{in } D_e \cup D_1 \cup D_2 \\ u^+ &= u^- & \text{on } \partial D_i, i = 1, 2 \\ \partial_{\nu_i}^+ u &= k \partial_{\nu_i}^- u & \text{on } \partial D_i, i = 1, 2. \\ |u|(x) &\rightarrow 0 & \text{as } |x| \rightarrow \infty \end{cases}$$

where $k = -\left(\frac{1+2\lambda}{1-2\lambda}\right) < 0$.

Conversely, if $u \in W^{1,-1}(\mathbb{R}^2)$ satisfies (2.4), and if $\psi_i := (\partial_{\nu_i} u^+ - \partial_{\nu_i} u^-)|_{\partial D_i}$, $i = 1, 2$, then $\psi = (\psi_1, \psi_2)$ is an eigenvector of K^* associated to λ .

3. The spectrum of the Neumann-Poincaré operator for a pair of close-to-touching discs

To compute the spectrum of K^* , we define

$$(3.1) \quad a = \sqrt{\delta(\delta+2)} \quad \rho = \frac{a-\delta}{a+\delta},$$

and we transform the close-to-touching discs via the conformal map

$$x = x_1 + ix_2 \quad \longrightarrow \quad \xi = \frac{x-a}{x+a}.$$

The disc D_1 is mapped into the disc $B(0, \rho)$, while D_2 is mapped into the complementary of the disc $B(0, \rho^{-1})$ (see Figure 1). Using Proposition (2.1), we seek

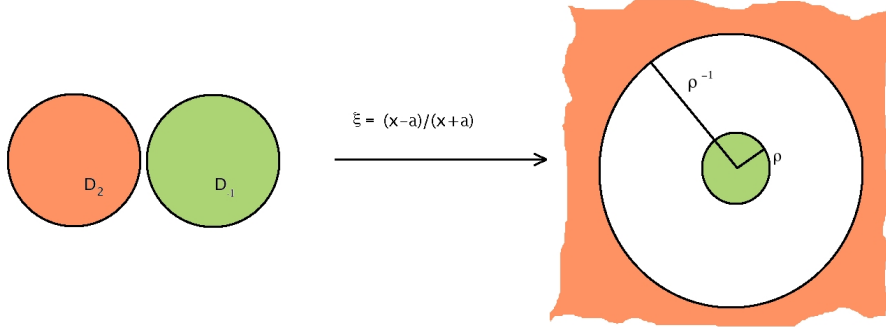


FIGURE 1. The conformal map

a non-trivial function u solution to (2.4) as the real part of a function which is harmonic on each component:

$$f(\xi) = \begin{cases} f_1(\xi) = \sum_{n \geq 0} (A_n^1 + iB_n^1) \xi^n, & \text{if } |\xi| < \rho \\ f_2(\xi) = \sum_{n \geq 0} (A_n^2 + iB_n^2) \xi^{-n}, & \text{if } |\xi| > \rho^{-1} \\ f_M(\xi) = \sum_{n \geq 0} (a_n^1 + ib_n^1) \xi^n + \sum_{n < 0} (a_n^2 + ib_n^2) \xi^{-n}, & \text{if } \rho < |\xi| < \rho^{-1}, \end{cases}$$

where the coefficients $a_n^i, b_n^i, A_n^i, B_n^i$ are real numbers to be determined. The function f should satisfy the transmission conditions

$$\begin{cases} \operatorname{Re}(f_1) + ik\operatorname{Im}(f_1) &= \operatorname{Re}(f_M) + i\operatorname{Im}(f_M) & \text{on } |\xi| = \rho \\ \operatorname{Re}(f_2) + ik\operatorname{Im}(f_2) &= \operatorname{Re}(f_M) + i\operatorname{Im}(f_M) & \text{on } |\xi| = \rho^{-1} \end{cases}$$

Expliciting these conditions when $\xi = \rho e^{in\theta}$ or when $\xi = \rho^{-1} e^{in\theta}$, we obtain for $n > 0$

$$\begin{pmatrix} 1 & 0 & -1 & -\rho^{-2n} \\ k & 0 & -1 & \rho^{-2n} \\ 0 & 1 & -\rho^{-2n} & -1 \\ 0 & k & \rho^{-2n} & -1 \end{pmatrix} \begin{pmatrix} A_n^1 \\ A_n^2 \\ a_n^1 \\ a_n^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 & \rho^{-2n} \\ k & 0 & -1 & -\rho^{-2n} \\ 0 & 1 & \rho^{-2n} & -1 \\ 0 & k & -\rho^{-2n} & -1 \end{pmatrix} \begin{pmatrix} B_n^1 \\ B_n^2 \\ b_n^1 \\ b_n^2 \end{pmatrix} = 0,$$

whereas when $n = 0$, the transmission conditions yield

$$\begin{cases} A_0^1 &= a_0^1 &= A_0^2, & \text{on } |\xi| = \rho \\ kB_0^1 &= b_0^1 &= kB_0^2, & \text{on } |\xi| = \rho^{-1}. \end{cases}$$

The condition that solutions to (2.4) satisfy $u(x) \rightarrow 0$ as $|x| \rightarrow \infty$ implies that $f(1) = 0$ and so $a_0^1 = b_0^1 = 0$. It follows that there is no eigenmode associated with $n = 0$. The above matrices have the same determinant, which vanishes for

$$\begin{aligned} k &= k_n^- &= -\frac{1 - \rho^{2n}}{1 + \rho^{2n}} \in (-1, 0) \\ \text{or } k &= k_n^+ &= -\frac{1 + \rho^{2n}}{1 - \rho^{2n}} \in (-\infty, -1). \end{aligned}$$

The associated eigenvectors split in A-modes and B-modes, corresponding to functions u which are even or odd in the ξ -plane. From Proposition 2.1, we deduce

PROPOSITION 3.1. In addition to the eigenvalue $1/2$, the spectrum of K^* is composed of the two families

$$\lambda_n^- = \frac{-\rho^{2n}}{2} \in (-1/2, 0), \quad \lambda_n^+ = \frac{\rho^{2n}}{2} \in (0, 1/2), \quad n > 0.$$

Each eigenvalue has multiplicity two, and is associated with an A mode and a B mode. The corresponding eigenspace is spanned by

$$\begin{aligned} \varphi_{n,A,\pm} &= \frac{|1 - \xi|^2}{2a} \cos(n\theta) \begin{pmatrix} \pm 2n\rho^{-n-1} \\ -2n\rho^{-n+1} \end{pmatrix} \\ \varphi_{n,B,\pm} &= \frac{|1 - \xi|^2}{2a} \sin(n\theta) \begin{pmatrix} \mp 2n\rho^{-n-1} \\ 2n\rho^{-n+1} \end{pmatrix}, \end{aligned}$$

for the A and B-modes respectively, where we write $\xi = \frac{z-a}{z+a} = \rho e^{i\theta}$. To each mode is associated a function $U_{n,A,\pm}$ or $U_{n,B,\pm}$ solution to (2.4), which in the ξ variables writes

$$\begin{aligned} U_{n,A,\pm}(\xi) &= \begin{cases} (1 \mp \rho^{-2n})r^n \cos(n\theta) & \text{if } r < \rho \\ (r^n \mp r^{-n}) \cos(n\theta) & \text{if } \rho < r < \rho^{-1} \\ \mp(1 \mp \rho^{-2n})r^n \cos(n\theta) & \text{if } \rho^{-1} < r \end{cases} \\ U_{n,B,\pm}(\xi) &= \begin{cases} (1 \mp \rho^{-2n})r^n \sin(n\theta) & \text{if } r < \rho \\ (r^n \pm r^{-n}) \sin(n\theta) & \text{if } \rho < r < \rho^{-1} \\ \pm(1 \mp \rho^{-2n})r^n \sin(n\theta) & \text{if } \rho^{-1} < r \end{cases} \end{aligned}$$

REMARK 3.2. As $\delta \rightarrow 0$, the eigenvalues λ_n^\pm converge to $\pm 1/2$ as illustrated in Fig. 2. This behavior shows the non-uniform compactness of the cross terms L_1, L_2 of K^* as $\delta \rightarrow 0$.

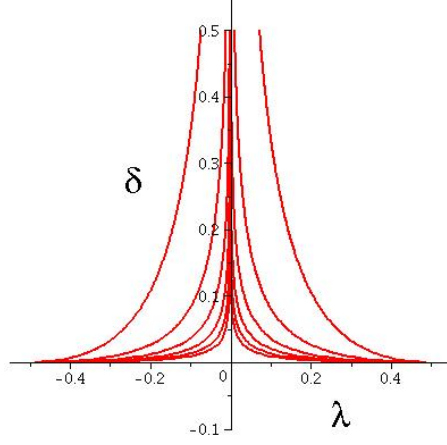


FIGURE 2. Graph of the eigenvalues λ_n^\pm in terms of δ

The results of [12] show that the eigenfunctions $\varphi_{n,A,\pm}, \varphi_{n,B,\pm}$ form a complete set in $H^{-1/2}(\partial D_1) \times H^{-1/2}(\partial D_2)$ for the norm $\|\cdot\|_S := \langle \cdot, \cdot \rangle_S^{1/2}$. The right-hand side $\partial_\nu H$ of (1.6) expands as

$$(\partial_\nu H|_{\partial D_1}, \partial_\nu H|_{\partial D_2}) = \sum_{n=1}^{\infty} \alpha_{n,\pm} \varphi_{n,A,\pm} + \beta_{n,\pm} \varphi_{n,B,\pm},$$

where $\alpha_{n,\pm} = \frac{\langle \varphi_{n,A,\pm}, \partial_\nu H \rangle_S}{\langle \varphi_{n,A,\pm}, \varphi_{n,A,\pm} \rangle_S}$ and $\beta_{n,\pm} = \frac{\langle \varphi_{n,B,\pm}, \partial_\nu H \rangle_S}{\langle \varphi_{n,B,\pm}, \varphi_{n,B,\pm} \rangle_S}$. Recalling (??), we note that since H is harmonic,

$$\begin{aligned} \langle \varphi_0, \partial_\nu H \rangle_S &= - \int_{\partial D_1} w \partial_\nu H - \int_{\partial D_2} w \partial_\nu H \\ &= -c_1 \int_{\partial D_1} \partial_\nu H - c_2 \int_{\partial D_2} \partial_\nu H = 0, \end{aligned}$$

and thus the right-hand side has no component on the eigenvector associated to $\lambda = 1/2$. It follows that the layer potential defined by

$$\varphi = \sum_{n=1}^{\infty} \frac{\alpha_{n,\pm}}{\lambda - \lambda_n^\pm} \varphi_{n,A,\pm} + \frac{\beta_{n,\pm}}{\lambda - \lambda_n^\pm} \varphi_{n,B,\pm}$$

is the solution of the integral equation (1.6).

The coefficients $\alpha_{n,\pm}, \beta_{n,\pm}$ in the above expansion can be computed explicitly. For example, in the case of A+ modes (i.e. modes of the A-family associated with

λ_n^+) we obtain

$$\begin{aligned}
\langle \varphi_{n,A,+}, \varphi_{n,A,+} \rangle_S &= - \int_{\partial D_1} S \varphi_{n,A,+} |_{\partial D_1} \varphi_{n,A,+} - \int_{\partial D_2} S \varphi_{n,A,+} |_{\partial D_2} \varphi_{n,A,+} \\
&= \int_{\mathbb{R}^2} |\nabla U_{n,A,+}|^2 \\
&= \int_{r=\rho} U_{n,A,+} (\partial_r U_{n,A,+}|^- - \partial_r U_{n,A,+}|^+) \\
&\quad - \int_{r=\rho^{-1}} U_{n,A,+} (\partial_r U_{n,A,+}|^- - \partial_r U_{n,A,+}|^+) \\
&= 4\pi n (\rho^{-2n} - 1). \\
\langle \varphi_{n,A,+}, \partial_\nu H \rangle_S &= - \int_{\partial D_1} S \varphi_{n,A,+} |_{\partial D_1} \partial_\nu H - \int_{\partial D_2} S \varphi_{n,A,+} |_{\partial D_2} \partial_\nu H \\
&= - \int_{r<\rho} \nabla U_{n,A,+} \cdot \nabla h - \int_{r>\rho^{-1}} \nabla U_{n,A,+} \cdot \nabla h \\
&= - \int_{r=\rho} \partial_r U_{n,A,+}|^- h + \int_{r=\rho^{-1}} \partial_r U_{n,A,+}|^- h \\
&= n\rho^n (\rho^{-2n} - 1) \left\{ \int_0^{2\pi} \cos(n\theta) [h(\rho e^{i\theta}) d\theta - h(\rho^{-1} e^{i\theta})] d\theta \right\},
\end{aligned}$$

where we have set $H(x) = h(\xi) = h(\rho e^{i\theta})$. Carrying out the computations for all the modes, one obtains

$$\begin{aligned}
\alpha_{n,\pm} &= \frac{1}{4\pi} \rho^n \left\{ \int_0^{2\pi} \cos(n\theta) [\pm h(\rho e^{i\theta}) - h(\rho^{-1} e^{i\theta})] d\theta \right\} \\
\beta_{n,\pm} &= \frac{1}{4\pi} \rho^n \left\{ \int_0^{2\pi} \sin(n\theta) [\mp h(\rho e^{i\theta}) + h(\rho^{-1} e^{i\theta})] d\theta \right\}
\end{aligned}$$

4. Asymptotics of the layer potential

In the particular case when $H(x) = Ax_1 + Bx_2$ is a linear function, we obtain

$$\alpha_{n,+} = -Aa\rho^{2n}, \quad \alpha_{n,-} = \beta_{n,+} = 0, \quad \beta_{n,-} = Ba\rho^{2n},$$

which yields the following expression for the layer potential

$$(4.1) \quad \varphi = |1 - \xi|^2 \sum_{n=1}^{\infty} \left\{ (-A) \frac{n \cos(n\theta)}{\lambda - \lambda_n^+} \begin{pmatrix} \rho^{n-1} \\ \rho^{n+1} \end{pmatrix} + B \frac{n \sin(n\theta)}{\lambda - \lambda_n^-} \begin{pmatrix} \rho^{n-1} \\ \rho^{n+1} \end{pmatrix} \right\}$$

When $\delta > 0$ and thus $\rho < 1$, it is easy to check that the above series converges pointwise, and not only in the sense of the norm associated with the scalar product defined by (??). In this section, we show the following result.

THEOREM 4.1. *Assume that H is the linear function $Ax_1 + Bx_2$. Then the layer potential φ solution to (1.6) satisfies the bound*

$$(4.2) \quad \|\varphi\|_{L^\infty(\partial D_1 \cup \partial D_2)} \leq \frac{C|A|}{|\lambda - \lambda_1^+|} + \frac{C|B|}{|\lambda - \lambda_1^-|},$$

where the constant C is independent of δ and λ .

PROOF. Here we only focus on the part of series (4.1) that blows up when λ is close to $1/2$ (i.e. the A+ modes)

$$(4.3) \quad \varphi_{A,+}(\theta) = -2A |1 - \rho e^{i\theta}|^2 \sum_{n=1}^{\infty} \frac{n\rho^{n-1} \cos(n\theta)}{2\lambda - \rho^{2n}}.$$

The remainder term of the series (4.1) only blows up when λ is close to $-1/2$ and can be treated in a similar way.

For $\lambda > \frac{1}{2}$, we have

$$\frac{1}{2\lambda - \rho^{2n}} = \frac{1}{2\lambda} \sum_{p=0}^{\infty} \left(\frac{\rho^{2n}}{2\lambda}\right)^p.$$

When $\delta > 0$, the series (4.3) converges uniformly on $[0, 2\pi]$, and so the order summation can be changed to obtain

$$(4.4) \quad \varphi_{A,+}(\theta) = -\frac{A}{\lambda} |1 - \rho e^{i\theta}|^2 \sum_{p=0}^{\infty} \left(\frac{1}{2\lambda}\right)^p \sum_{n=1}^{\infty} n (\rho^{2p+1})^n \cos(n\theta).$$

A straightforward computation shows that

$$\sum_{n=1}^{\infty} nr^n \cos(n\theta) = \operatorname{Re} \left(\frac{re^{i\theta}}{(1 - re^{i\theta})^2} \right) \quad \text{for } r < 1.$$

It follows that

$$|\varphi_{A,+}(\theta)| \leq \frac{\rho|A|}{\lambda} \sum_{p=0}^{\infty} \left(\frac{\rho^2}{2\lambda}\right)^p \left| \frac{1 - \rho e^{i\theta}}{1 - \rho^{2p+1} e^{i\theta}} \right|^2.$$

Using the fact that

$$\left| \frac{1 - \rho e^{i\theta}}{1 - \rho^{2p+1} e^{i\theta}} \right|^2 \leq 1 + \rho,$$

uniformly with respect to θ and $p \geq 0$, we get

$$|\varphi_{A,+}(\theta)| \leq \frac{(\rho + \rho^2)}{\lambda} |A| \sum_{p=0}^{\infty} \left(\frac{\rho^2}{2\lambda}\right)^p.$$

Consequently

$$|\varphi_{A,+}(\theta)| \leq (\rho + \rho^2) \frac{|A|}{\lambda - \frac{\rho^2}{2}},$$

which gives the first bound in the desired inequality. \square

REMARK 4.2. 1. Since $\rho = 1 - \sqrt{2}\sqrt{\delta} + O(\delta)$ and in view of the definition of λ_1^\pm , it is easy to check that the blow up rate of φ is the same as that of u in (1.4,1.5). In fact, since $u = S_1\varphi|_{\partial D_1} + S_2\varphi|_{\partial D_2} + H$, since ∂D_1 and ∂D_2 are smooth, and since the single layer operators S_1 and S_2 depend smoothly on δ , the $W^{1,\infty}$ control of u follows from (4.2).

2. In this section we only considered harmonic functions H which are linear. Indeed, one can show that the blow up of φ may only occur when $\nabla H(0) \neq 0$.

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