Morfismos

Departamento de Matemáticas
Cinvestav

Morfismos, Volumen 17, Número 2, julio a diciembre 2013, es una publicación semestral editada por el Centro de Investigación y de Estudios Avanzados del Instituto Politécnico Nacional (**Cinvestav**), a través del Departamento de Matemáticas. Av. Instituto Politécnico Nacional No. 2508, Col. San Pedro Zacatenco, Delegación Gustavo A. Madero, C.P. 07360, D.F., Tel. 55-57473800, www.cinvestav.mx, morfismos@math.cinvestav.mx, Editores Generales: Drs. Isidoro Gitler y Jesús González Espino Barros. Reserva de Derechos No. 04-2012-011011542900-102, ISSN: 1870-6525, ambos otorgados por el Instituto Nacional del Derecho de Autor. Certificado de Licitud de Título No. 14729, Certificado de Licitud de Contenido No. 12302, ambos otorgados por la Comisión Calificadora de Publicaciones y Revistas Ilustradas de la Secretaría de Gobernación. Impreso por el Departamento de Matemáticas del Cinvestav, Avenida Instituto Politécnico Nacional 2508, Colonia San Pedro Zacatenco, C.P. 07360, México, D.F. Este número se terminó de imprimir en febrero de 2014 con un tiraje de 50 ejemplares.

Morfismos

Departamento de Matemáticas
Cinvestav

Morfismos está disponible en la dirección *http://www.morfismos.cinvestav.mx*. Para mayores informes dirigirse al teléfono +52 (55) 5747-3871. Toda correspondencia debe ir dirigida a la Sra. Laura Valencia, Departamento de Matemáticas del Cinvestav, Apartado Postal 14-740, México, D.F. 07000, o por correo electrónico a la dirección: *morfismos@math.cinvestav.mx*.

Information for Authors

The Editorial Board of *Morfismos* calls for papers on mathematics and related areas to be submitted for publication in this journal under the following guidelines:

- Manuscripts should fit in one of the following three categories: (a) papers covering the graduate work of a student, (b) contributed papers, and (c) invited papers by leading scientists. Each paper published in *Morfismos* will be posted with an indication of which of these three categories the paper belongs to.
- Papers in category (a) might be written in Spanish; all other papers proposed for publication in *Morfismos* shall be written in English, except those for which the Editoral Board decides to publish in another language.
- All received manuscripts will be refereed by specialists.
- In the case of papers covering the graduate work of a student, the author should provide the supervisor's name and affiliation, date of completion of the degree, and institution granting it.
- Authors may retrieve the LaTeX macros used for *Morfismos* through the web site *http://www.math.cinvestav.mx*, at "Revista Morfismos". The use by authors of these macros helps for an expeditious production process of accepted papers.
- All illustrations must be of professional quality.
- Authors will receive the pdf file of their published paper.
- Manuscripts submitted for publication in *Morfismos* should be sent to the email address *morfismos@math.cinvestav.mx*.

Información para Autores

El Consejo Editorial de *Morfismos* convoca a proponer artículos en matemáticas y áreas relacionadas para ser publicados en esta revista bajo los siguientes lineamientos:

- Se considerarán tres tipos de trabajos: (a) artículos derivados de tesis de grado de alta calidad, (b) artículos por contribución y (c) artículos por invitación escritos por líderes en sus respectivas áreas. En todo artículo publicado en *Morfismos* se indicará el tipo de trabajo del que se trate de acuerdo a esta clasificación.
- Los artículos del tipo (a) podrán estar escritos en español. Los demás trabajos deberán estar redactados en inglés, salvo aquellos que el Comité Editorial decida publicar en otro idioma.
- Cada artículo propuesto para publicación en *Morfismos* será enviado a especialistas para su arbitraje.
- En el caso de artículos derivados de tesis de grado se debe indicar el nombre del supervisor de tesis, su adscripción, la fecha de obtención del grado y la institución que lo otorga.
- Los autores interesados pueden obtener el formato LaTeX utilizado por *Morfismos* en el enlace "Revista Morfismos" de la dirección *http://www.math.cinvestav.mx*. La utilización de dicho formato ayudará en la pronta publicación de los artículos aceptados.
- Si el artículo contiene ilustraciones o figuras, éstas deberán ser presentadas de forma que se ajusten a la calidad de reproducción de *Morfismos*.
- Los autores recibirán el archivo pdf de su artículo publicado.
- Los artículos propuestos para publicación en *Morfismos* deben ser dirigidos a la dirección *morfismos@math.cinvestav.mx*.

Editorial Guidelines

Morfismos is the journal of the Mathematics Department of Cinvestav. One of its main objectives is to give advanced students a forum to publish their early mathematical writings and to build skills in communicating mathematics.

Publication of papers is not restricted to students of Cinvestav; we want to encourage students in Mexico and abroad to submit papers. Mathematics research reports or summaries of bachelor, master and Ph.D. theses of high quality will be considered for publication, as well as contributed and invited papers by researchers. All submitted papers should be original, either in the results or in the methods. The Editors will assign as referees well-established mathematicians, and the acceptance/rejection decision will be taken by the Editorial Board on the basis of the referee reports.

Authors of *Morfismos* will be able to choose to transfer copy rights of their works to *Morfismos*. In that case, the corresponding papers cannot be considered or sent for publication in any other printed or electronic media. Only those papers for which *Morfismos* is granted copyright will be subject to revision in international data bases such as the American Mathematical Society's Mathematical Reviews, and the European Mathematical Society's Zentralblatt MATH.

Morfismos

Lineamientos Editoriales

Morfismos, revista semestral del Departamento de Matemáticas del Cinvestav, tiene entre sus principales objetivos el ofrecer a los estudiantes más adelantados un foro para publicar sus primeros trabajos matemáticos, a fin de que desarrollen habilidades adecuadas para la comunicación y escritura de resultados matemáticos.

La publicación de trabajos no está restringida a estudiantes del Cinvestav; deseamos fomentar la participación de estudiantes en México y en el extranjero, así como de investigadores mediante artículos por contribución y por invitación. Los reportes de investigación matemática o resúmenes de tesis de licenciatura, maestría o doctorado de alta calidad pueden ser publicados en *Morfismos*. Los artículos a publicarse serán originales, ya sea en los resultados o en los métodos. Para juzgar ésto, el Consejo Editorial designará revisores de reconocido prestigio en el orbe internacional. La aceptación de los artículos propuestos será decidida por el Consejo Editorial con base a los reportes recibidos.

Los autores que así lo deseen podrán optar por ceder a *Morfismos* los derechos de publicación y distribución de sus trabajos. En tal caso, dichos artículos no podrán ser publicados en ninguna otra revista ni medio impreso o electrónico. *Morfismos* solicitará que tales artículos sean revisados en bases de datos internacionales como lo son el Mathematical Reviews, de la American Mathematical Society, y el Zentralblatt MATH, de la European Mathematical Society.

Morfismos

Mediterranean Institute for the Mathematical Sciences

Special MIMS Proceedings Issue

This issue is devoted to the proceedings of the conference "Operads and Configuration Spaces" that took place at the Mediterranean Institute for the Mathematical Sciences (MIMS), Cité des Sciences, in Tunis capital city, June 18–22, 2012. This conference was part of the launch of MIMS in the region. Plenary speakers gave a series of lectures which were attended by students and young researchers from Tunisia and Algeria. The MIMS thanks Christophe Cazanave, Jeffrey Giansiracusa, Paolo Salvatore, Ines Saihi, Ismar Volić, Benjamin Walter, and all participants for making this a successful first conference. It also thanks Oscar-Randal Williams for his special contribution.

Este número está dedicado a las memorias de la conferencia "Operads and Configuration Spaces" realizada en el Mediterranean Institute for Mathematical Sciences (MIMS), Cité des Sciences, en la ciudad de Túnez, del 18 al 22 de junio de 2012. La conferencia fue parte de las actividades inaugurales del MIMS en la región. Los ponentes plenarios dieron una serie de conferencias a las que asistieron estudiantes e investigadores jóvenes de Túnez y Argelia. El MIMS agradece a Christophe Cazanave, Jeffrey Giansiracusa, Paolo Salvatore, Ines Saihi, Ismar Volić, Benjamin Walter y todos los participantes por hacer de esta primera conferencia un éxito. También agradece a Oscar-Randal Williams por su contribución especial.

Contents - Contenido

Morfismos, Vol. 17, No. 2, 2013, pp. 1–56

Configuration space integrals and the topology of knot and link spaces

Ismar Volić [1]

Abstract

This article surveys the use of configuration space integrals in the study of the topology of knot and link spaces. The main focus is the exposition of how these integrals produce finite type invariants of classical knots and links. More generally, we also explain the construction of a chain map, given by configuration space integrals, between a certain diagram complex and the deRham complex of the space of knots in dimension four or more. A generalization to spaces of links, homotopy links, and braids is also treated, as are connections to Milnor invariants, manifold calculus of functors, and the rational formality of the little balls operads.

2010 Mathematics Subject Classification: 57Q45, 57M27, 81Q30, 57-R40.
Keywords and phrases: configuration space integrals, Bott-Taubes integrals, knots, links, homotopy links, braids, finite type invariants, Vassiliev invariants, Milnor invariants, chord diagrams, weight systems, manifold calculus, embedding calculus, little balls operad, rational formality of configuration spaces.

Contents

[1]The author was supported by the National Science Foundation grant DMS 1205786.

1 Introduction

Configuration space integrals are fascinating objects that lie at the intersection of physics, combinatorics, topology, and geometry. Since their inception over twenty years ago, they have emerged as an important tool in the study of the topology of spaces of embeddings and in particular of spaces of knots and links.

The beginnings of configuration space integrals can be traced back to Guadagnini, Martellini, and Mintchev [19] and Bar-Natan [4] whose work was inspired by Chern-Simons theory. The more topological point of view was introduced by Bott and Taubes [9]; configuration space integrals are because of this sometimes even called *Bott-Taubes integrals* in the literature (more on Bott and Taubes' work can be found in Section 3.3). The point of this early work was to use configuration space integrals to construct a knot invariant in the spirit of the classical linking number of a two-component link. This invariant turned out to be of *finite type* (finite type invariants are reviewed in Section 2.3) and D. Thurston [51] generalized it to construct all finite type invariants. We will explain D. Thurston's result in Section 3.4, but the idea is as follows:

Given a trivalent diagram Γ (see Section 2.3), one can construct a bundle

$$\pi\colon \operatorname{Conf}[p,q;\mathcal{K}^3,\mathbb{R}^n] \longrightarrow \mathcal{K}^3,$$

where $\mathcal{K}^3$ is the space of knots in $\mathbb{R}^3$. Here p and q are the numbers of certain kinds of vertices in Γ and $\mathrm{Conf}[p, q; \mathcal{K}^3, \mathbb{R}^n]$ is a pullback space constructed from an evaluation map and a projection map. The fiber of π over a knot $K \in \mathcal{K}^3$ is the compactified configuration space of $p + q$ points in $\mathbb{R}^3$, first p of which are constrained to lie on K. The edges of Γ also give a prescription for pulling back a product of volume forms on S^2 to $\mathrm{Conf}[p, q; \mathcal{K}^3, \mathbb{R}^n]$. The resulting form can then be integrated along the fiber, or pushed forward, to $\mathcal{K}^3$. The dimensions work out so that this is a 0-form and, after adding the pushforwards over all trivalent diagrams of a certain type, this form is in fact closed, i.e. it is an invariant. Thurston then proves that this is a finite type invariant and that this procedure gives all finite type invariants.

The next generalization was carried out by Cattaneo, Cotta-Ramusino, and Longoni [12]. Namely, let $\mathcal{K}^n$, $n > 3$, be the space of knots in $\mathbb{R}^n$. The main result of [12] is that there is a cochain map

$$\mathcal{D}^n \longrightarrow \Omega^*(\mathcal{K}^n) \tag{1}$$

between a certain diagram complex $\mathcal{D}^n$ generalizing trivalent diagrams and the deRham complex of $\mathcal{K}^n$. The map is given by exactly the same integration procedure as Thurston's, except the degree of the form that is produced on $\mathcal{K}^n$ is no longer zero. Specializing to classical knots (where there is no longer a cochain map due to the so-called "anomalous face"; see Section 3.4) and degree zero, one recovers the work of Thurston. Cattaneo, Cotta-Ramusino, and Longoni have used the map (1) to show that spaces of knots have cohomology in arbitrarily high degrees in [13] by studying certain algebraic structures on $\mathcal{D}^n$ that correspond to those in the cohomology ring of $\mathcal{K}^n$. Longoni also proved in [33] that some of these classes arise from non-trivalent diagrams.

Even though configuration space integrals were in all of the aforementioned work constructed for ordinary closed knots, it has in recent years become clear that the variant for long knots is also useful. Because some of the applications we describe here have a slight preference for the long version, this is the space we will work with. The difference between the closed and the long version is minimal from the perspective of this paper, as explained at the beginning of Section 2.2.

More recently, configuration space integrals have been generalized to (long) links, homotopy links, and braids [30, 57], and this work is summarized in Section 5.1. One nice feature of this generalization is that it provides the connection to Milnor invariants. This is because configuration space integrals give finite type invariants of homotopy links, and, since Milnor invariants are finite type, this immediately gives integral expressions for these classical invariants.

We also describe two more surprising applications of configuration space integrals. Namely, one can use *manifold calculus of functors* to place finite type invariants in a more homotopy-theoretic setting as described in Section 5.2. Functor calculus also combines with the *formality of the little n-discs operad* to give a description of the rational homology of $\mathcal{K}^n$, $n > 3$. Configuration space integrals play a central role here since they are at the heart of the proof of operad formality. Some details about this are provided in Section 5.3.

In order to keep the focus of this paper on knot and links and keep its length to a manageable size, we will regrettably only point the reader to three other topics that are growing in promise and popularity. The first is the work of Sakai [44] and its expansion by Sakai and Watanabe [49] on *long planes*, namely embeddings of $\mathbb{R}^k$ in $\mathbb{R}^n$ fixed outside a compact set. These authors use configuration space integrals to produce nontrivial cohomology classes of this space with certain conditions on k and n. This work generalizes classes produced by others [14, 58] and complements recent work by Arone and Turchin [2] who show, using homotopy-theoretic methods, that the homology of $\mathrm{Emb}(\mathbb{R}^k, \mathbb{R}^n)$ is given by a certain graph complex for $n \geq 2k+2$. Sakai has further used configuration space integrals to produce a cohomology class of $\mathcal{K}^3$ in degree one that is related to the Casson invariant [43] and has given a new interpretation of the Haefliger invariant for $\mathrm{Emb}(\mathbb{R}^k, \mathbb{R}^n)$ for some k and n [44]. In an interesting bridge between two different points of view on spaces of knots, Sakai has in [44] also combined the configuration space integrals with Budney's action of the little discs operad on $\mathcal{K}^n$ [10].

The other interesting development is the recent work of Koytcheff [29] who develops a homotopy-theoretic replacement of configuration space integrals. He uses the Pontryagin-Thom construction to "push forward" forms from $\mathrm{Conf}[p, q; \mathcal{K}^n, \mathbb{R}^n]$ to $\mathcal{K}^n$. The advantage of this approach is that is works over any coefficients, unlike ordinary configuration space integration, which takes values in $\mathbb{R}$. A better understanding of how Koytcheff's construction relates to the original configuration space integrals is still needed.

The third topic is the role configuration space integrals have recently played in the construction of asymptotic finite type invariants of divergence-free vector fields [25]. The approach in this work is to apply configuration space integrals to trajectories of a vector field. In this way, generalizations of some familiar asymptotic vector field invariants like asymptotic linking number, helicity, and the asymptotic signature can be derived.

Lastly, some notes on the style and expositional choices we have made in this paper are in order. We will assume an informal tone, especially at times when writing down something precisely would require us to introduce cumbersome notation. To quote from a friend and coauthor Brian Munson [40], "we will frequently omit arguments which would distract us from our attempts at being lighthearted". Whenever this is the case, a reference to the place where the details appear will be supplied. In particular, most of the proofs we present here have been worked out in detail elsewhere, and if we feel that the original source is sufficient, we will simply give a sketch of the proof and provide ample references for further reading. It is also worth pointing out that many open problems are stated througout and our ultimate hope is that, upon looking at this paper, the reader will be motivated to tackle some of them.

1.1 Organization of the paper

We begin by recall some of the necessary background in Section 2. We only give the basics but furnish abundant references for further reading. In particular, we review integration along the fiber in Section 2.1 and pay special attention to integration for infinite-dimensional manifolds and manifolds with corners. In Section 2.2 we define the space of long knots and state some observations about it. A review of finite type invariants is provided in Section 2.3; they will play a central role later. This section also includes a discussion of chord diagrams and trivalent diagrams. Finally in Section 2.4, we talk about configuration spaces and their Fulton-MacPherson compactification. These are the spaces over which our integration will take place.

Section 3 is devoted to the construction of finite type invariants via configuration space integrals. The motivating notion of the linking number is recalled in Section 3.1, and that leads to the failed construction of the "self-linking" number in Section 3.2 and its improvement to the simplest finite type (Casson) invariant in Section 3.3. This section is at

Morfismos, Vol. 17, No. 2, 2013, pp. 71–100

Cooperads as symmetric sequences

Benjamin Walter

Abstract

We give a brief overview of the basics of cooperad theory using a new definition which lends itself to easy example creation and verification and avoids common pitfalls and complications caused by nonassociativity of the composition operation for cooperads. We also apply our definition to build the parenthesization and cosimplicial structures exhibited by cooperads and give examples.

2010 Mathematics Subject Classification: 18D50; 16T15, 17B62.
Keywords and phrases: Cooperads, operads, coalgebras, Kan extensions.

1 Introduction

In the current work we discuss cooperads in generic symmetric monoidal categories from the point of view of symmetric sequences. Fix a symmetric monoidal category $(\mathcal{C}, \otimes)$. Let us roughly recall the standard framework.

Operads encode algebra structures. The tautological example is the endomorphism operad of an object $\mathrm{END}(A) = \coprod_n \mathrm{Hom}(A^{\otimes n}, A)$. Operads have a natural grading by levels expressing the "arity" of different "operations" (for example, $\mathrm{END}(A)(n) = \mathrm{Hom}(A^{\otimes n}, A)$). The symmetric group Σ_n acts on the n-ary operations of an operad (for $\mathrm{END}(A)(n)$ this action is by permutation of the $A^{\otimes n}$). A graded object with Σ_n-actions is called a "symmetric sequence." Operads are further equipped with a composition product identifying the result of plugging operations into each other (for example, $\mathrm{END}(A) \circ \mathrm{END}(A) \to \mathrm{END}(A)$). Very roughly, an operad is "a bunch of objects with a rule for plugging them into each other".

Operads encode algebra structures via maps of operads (preserving symmetric group actions and composition structure). So, for example, there is an operad $\textsc{lie}$ of formal Lie bracket expressions modulo Lie relations, along with a composition rule identifying the result of plugging bracket expressions into each other. A map of operads $\textsc{lie} \to \textsc{end}(A)$ identifies a specific endomorphism of A for each formal Lie bracket expression, in such a way that composition of Lie bracket expressions is compatible with the composition of corresponding endomorphisms. This gives A the structure of a Lie algebra.

Coalgebra structures can also be defined via operads. The coendomorphisms of an object $\textsc{coend}(A) = \coprod_n \operatorname{Hom}(A, A^{\otimes n})$ also form an operad: It is graded, with symmetric group action, and has a natural map $\textsc{coend}(A) \circ \textsc{coend}(A) \to \textsc{coend}(A)$ also given by plugging things into each other. Replacing $\textsc{end}$ by $\textsc{coend}$ changes algebra structures to coalgebra structures. For example a map of operads $\textsc{lie} \to \textsc{coend}(A)$ identifies a coendomorphism of A for each Lie bracket expression, thus giving A a Lie coalgebra structure.

This is a common point of view (see e.g. [11]), but there is an alternative. For clarity, we will continue with the example of Lie algebras. A Lie algebra structure is maps $\textsc{lie}(n) \to \operatorname{Hom}(A^{\otimes n}, A)$ which is equivalent to maps $\textsc{lie}(n) \otimes A^{\otimes n} \to A$ (ignore Σ_n-actions for the moment). Dually, a Lie coalgebra structure is maps $\textsc{lie}(n) \to \operatorname{Hom}(A, A^{\otimes n})$ which is equivalent to maps $\textsc{lie}(n) \otimes A \to A^{\otimes n}$ which is equivalent to $A \to \big(\textsc{lie}(n)\big)^* \otimes A^{\otimes n}$. (Dualizing $\textsc{lie}(n)$ should not introduce trouble, because it is finite dimensional.) The level-wise dual object $\textsc{lie}^{\vee} = \coprod_n \big(\textsc{lie}(n)\big)^*$ has structure dual to that of $\textsc{lie}$. This is a cooperad. (The precise definition is the subject of the current paper.)

Experience [9] [10] has shown that it is sometimes more useful to directly work with cooperads and cooperad structures when describing coalgebras rather than continually referring all the way back to operads and operad structures. Also sometimes coalgebras can have a more natural expression as coalgebras over cooperads, rather than coalgebras over operads. Just as operads can be thought of as "a bunch of objects which are plugged into each other", cooperads can be thought of as "a bunch of objects where subobjects are removed or quotiented".

Unfortunately category theory causes a slight hitch when attempting to blindly dualize operad structure to define cooperads. The dual of operad composition is cooperad composition, which is similar except for some colimits being replaced by limits. The problem comes when looking at associativity. In a symmetric monoidal category $\otimes$ is left adjoint (to Hom) so it will commute with colimits. This allows operad composition products to be associative (e.g. $(\textsc{lie}\circ\textsc{lie})\circ\textsc{lie} = \textsc{lie}\circ(\textsc{lie}\circ$

LIE)). However, this will generally not happen for cooperad composition (e.g. $(\text{LIE}^\vee \bullet \text{LIE}^\vee) \bullet \text{LIE}^\vee \neq \text{LIE}^\vee \bullet (\text{LIE}^\vee \bullet \text{LIE}^\vee)$). This issue crops up for example, in the cooperadic cobar constructions of Ching in his thesis [4] and arXiv note [5].

We work by defining a new composition product – a composition product of tree-functors. The motivating intuition is that *the composition product of two symmetric sequences should not itself be a symmetric sequence* – in particular its group of symmetries is much too large. Maps to and from the tree-functor composition product can be expressed as maps to and from universal extensions, which yields the classical operad and cooperad composition products. Using the tree-functor composition product (rather than its Kan extension) when describing or defining cooperads greatly simplifies bookkeeping; though it turns out that, for operads, it doesn't really make a difference.

We begin by introducing the notation of wreath product categories. These are inspired by the wreath product categories of Berger [2], and at the most basic level are merely Groethendieck constructions. Wreath product categories are defined so that they will be the natural source category for iterated composition products of symmetric sequences. We use this to give a simple definition of cooperads and prove all of the standard structure holds. Then we describe comodules and coalgebras. We finish with simple examples related to work in [9], [10], and [13].

In the sequel [12] we use the structure presented here to build cofree coalgebras, connecting to the constructions of Fox [6] and Smith [11].

We assume that the reader is comfortable with the category theory notions of adjoint functors and Kan extensions, as well as basic simplicial and cosimplicial structures. A familiarity with the classical definitions of operads and their modules/algebras is not required, but would be helpful.

2 Wreath product categories

This section is divided into two parts. In the first subsection, we define wreath product categories using functors to the category of finite sets. Our definition is related to, but more general than, the dual of refined partitions of sets as used in literature by e.g. Arone-Mahowald [1]. The salient difference between wreath categories and refined partitions is that wreath categories incorporate the empty-set (see Remark 2.8). In the second subsection, an equivalent definition is given in terms of

Morfismos, Vol. 17, No. 2, 2013, pp. 101–125

Moduli spaces and modular operads

Jeffrey Giansiracusa [1]

Abstract

We describe a generalised ribbon graph decomposition for a broad class of moduli spaces of geometric structures on surfaces (with nonempty boundary), including moduli of spin surfaces, r-spin surfaces, surfaces with a principle G-bundle, surfaces with maps to a background space, surfaces with Higgs bundle, etc.

2010 Mathematics Subject Classification: 57M50, 57M15, 18D50, 18D05, 58D27.
Keywords and phrases: Ribbon graphs, moduli spaces, mapping class group, arc complex, 2-categories, cyclic operads.

1 Introduction

This paper is an expansion of some ideas that I first talked about in 2012 in the MIMS conference on Operads and Configuration Spaces. Here I shall give a more detailed account, though still not a complete one, of a certain theorem about modular envelopes. The full details will appear in a future paper; in this note I will try to be expository and focus on illuminating the central ideas without being overly concerned by technical details that might otherwise obscure some of the conceptual clarity of the arguments.

Fix a class ψ of geometric structures on surfaces. For example, one could take orientations, principal G-bundles, or spin structures, etc. Associated to any surface Σ is the *space* $\psi(\Sigma)$ of all such structures on that surface. Taking the homotopy quotient by the diffeomorphism group yields a homotopy theoretic moduli space of surfaces with ψ-structure. If we consider surfaces with some marked intervals along the boundary, and ψ-structures that have a fixed value on each marked interval, then we can glue the intervals together and the result is a modular operad.

denoted $\mathcal{M}_\psi$. (If, instead of a single fixed value on the intervals, we allow one of several fixed valued then the result is instead a coloured modular operad). These moduli spaces are the objects we wish to study. The idea of this work is to decompose them, in a sense, into moduli spaces of discs with ψ-structure. The modular operad $\mathcal{M}_\psi$ contains a sub-cyclic operad $\mathcal{D}_\psi$ of moduli spaces of discs with ψ-structure. Our main result is that $\mathcal{M}_\psi$ is freely generated (in a homotopical sense) as a modular operad over this sub-cyclic operad. I.e., the derived modular envelope of $\mathcal{D}_\psi$ is weakly equivalent to $\mathcal{M}_\psi$.

This result was inspired by the work of Costello. As part of his groundbreaking work in the homotopy theory of open-closed topological field theories [9], he gave a new perspective on the very important idea of describing the moduli space of Riemann surfaces with ribbon graphs in [8, 10]. He proved that the derived (i.e., homotopy invariant) modular envelope of the associative operad gives a model for the modular operad of moduli spaces of Riemann surfaces with open-string type gluing for the compositions. A point in this modular envelope can be described as a graph equipped with lengths on all of its edges and a cyclic order of the edges incident at each vertex — i.e., a metric ribbon graph. Thus the moduli space of ribbon graphs is equivalent to the moduli space of Riemann surfaces.

Costello's proof used geometry and analysis on a certain partial compactification of the moduli space of Riemann surfaces. Thus it appears his argument is not suited to more homotopy theoretic contexts such as the one considered in this paper. In [13], I gave a different proof of Costello's modular envelope theorem. This proof instead rested on the well-known contractibility of the arc complex of a surface. This new argument lead to an adaptation to dimension 3: the derived modular envelope of the framed little 2-discs is equivalent to the modular operad of moduli spaces of 3-dimensional handlebodies.

Here we instead focus of refining and generalising the argument of [13] in dimension 2. When the structures being considered are principal G-bundles then we expect this result will lead to a G-equivariant version of Costello's open-closed TFT theorem.

2 Operads

A *cyclic operad* in $\mathscr{C}$ is a functor $\mathcal{P}$ from the category of finite sets and bijections to $\mathscr{C}$ together with composition maps

$$\mathcal{P}(I) \otimes P(J) \stackrel{{}_i\circ_j}{\to} \mathcal{P}(I \sqcup J \smallsetminus \{i,j\}),$$

for $i \in I$ and $j \in J$, satisfying an associativity condition and natural in (I, i) and (J, j). One can think of $\mathcal{P}$ as a collection of abstract "electrical circuit components," where $\mathcal{P}(I)$ as a set/space of components with terminals given by the set I. The composition maps correspond to wiring terminals together to produce new components; terminals can only be glued in pairs (no trivalent connections) and in a cyclic operad two components can only be glued together in at most one place. Allowing multiple gluings leads to the following definition.

A *modular operad* in $\mathscr{C}$ is a cyclic operad $\mathcal{Q}$ together with natural self-composition maps $\mathcal{Q}(I) \stackrel{\circ_{i,j}}{\to} \mathcal{Q}(I \smallsetminus \{i,j\})$ that commute with the cyclic operad composition maps and with each other.

Example 2.1. 1. The commutative modular operad is the constant functor sending each finite set to a point.

2. The associative cyclic operad $\mathcal{A}ssoc$ sends I to the set of cyclic orders on I.

We will need a slight generalisation in which there are different types of terminals and two terminals can only be connected if they are of the same type. The types are called *colours*. Fix a set Λ, which we will call the set of colours. A Λ-coloured set I consists of a finite set with a map to Λ. A morphism $I \to I'$ of coloured sets is a bijection that respects the colours. A *coloured cyclic operad* $\mathcal{P}$ is a functor from the category of coloured sets to $\mathscr{C}$ together with a collection of composition maps ${}_i\circ_j$ as before, but now only defined when i and j have the same colour. A *coloured modular operad* is defined analogously, where the self-composition maps $\circ_{ij}$ also only defined when i and j have the same colour.

2.1 Homotopy theory of cyclic and modular operads

Berger and Batanin [7] have recently constructed fully satisfactory Quillen model category structures on cyclic and modular operads. When talking about derived constructions such as the derived modular envelope, one could work with the model category structures. However, we take a more pragmatic approach, since the modular envelope is the only functor we ever have to derive, and our construction of the derived functor will be manifestly homotopy invariant due to the homotopy invariance of